Part D: On-Chip Networks (24 points + 10 bonus)

You are choosing between two topologies for your on-chip network, shown below.

For a system of $N$ nodes, the 2-D Torus topology consists of $k = \sqrt{N}$ rows and columns, where each row or column is a ring.

**Question 1 (10 points)**

Your first task is to compare the topologies along key metrics. Fill in the table below as a function of the number of nodes in the network, $N$. The units for each cell are hops or links. For average distance, assume uniform random traffic (where each node sends $1/N^{th}$ of the traffic to each destination, including itself).

To ease your derivations, you can define a variable $k = \sqrt{N}$, and assume $k$ is an even integer.

*For partial credit, give the asymptotic growth instead.*

<table>
<thead>
<tr>
<th></th>
<th>Ring</th>
<th>2-D Torus</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number of links</strong></td>
<td>$N$</td>
<td>$2N$</td>
</tr>
<tr>
<td><strong>Diameter</strong></td>
<td>$\frac{N}{2}$</td>
<td>$\sqrt{N}$</td>
</tr>
<tr>
<td><strong>Average distance</strong></td>
<td>$\frac{N}{4}$</td>
<td>$\frac{\sqrt{N}}{2}$</td>
</tr>
<tr>
<td><strong>Bisection bandwidth</strong></td>
<td>$2$</td>
<td>$2\sqrt{N}$</td>
</tr>
</tbody>
</table>
**Question 2 (8 points)**

As discussed in lecture, higher-dimensional topologies have smaller diameter, but the routers are more expensive. For what range of number of nodes, $N$, is the average latency in cycles of a 2-D torus topology lower than the average latency in cycles of a ring topology? (Provide an inequality on $N$.)

Assume the following:
- Traversing one link requires 1 cycle
- Traversing one router requires 1 cycles in the ring topology
- Traversing one router requires 4 cycles in the 2-D torus ring topology
- Entering and exiting the network at a router counts as traversing the router
- Traffic is uniform random

(For example, in a 4-node ring, the latency to send a flit from a node to a node two hops away is 5 cycles: 1 cycle to enter the network at the first router, 2 cycles for two link traversals, 1 cycle for one router traversal, and 1 cycle to exit the network at the last router.)

Consider the Ring topology.
Every hop takes 2 cycles (1 for link, 1 for router, including the exit router).
The average distance in hops for uniform random traffic is $N/4$.
Add 1 cycle to enter the network.
$L_{\text{ring}} = N/4 \times 2 + 1 = N/2 + 1$

Consider the 2-D Torus topology.
Every hop takes 5 cycles (1 for link, 4 for router)
Average distance in hops is $\sqrt{N}/2$.
Add 4 cycles to enter the network.
$L_{\text{torus}} = \sqrt{N}/2 \times 5 + 4 = 5/2 \sqrt{N} + 4$

When is $L_{\text{torus}} < L_{\text{ring}}$?
=> $5/2 \sqrt{N} + 4 < N/2 + 1$
=> $0 < N - 5\sqrt{N} - 6$
=> $0 < (\sqrt{N} - 6)(\sqrt{N} + 1)$
=> $N > 36$

Since $N$ is an even square, $N \geq 64$.  

**Question 3 (6 points)**

In a stroke of genius, you realize that a Hierarchical Ring topology (shown below) can bridge the low-latency routing of a Ring topology with the lower diameter of a 2-D Torus topology. For a system of $N$ nodes, the Hierarchical Ring topology consists of one *global* ring surrounded by $k = \sqrt{N}$ *local rings*. Each local ring has one node that also bridges with the global ring.

Fill in the table below as a function of the number of nodes in the network, $N$. The units for each cell are hops or links. To ease your derivations, you can define a variable $k = \sqrt{N}$, and assume $k$ is an even integer. *For partial credit, give the asymptotic growth instead.*

![Hierarchical Ring Topology](image)

<table>
<thead>
<tr>
<th></th>
<th>Hierarchical Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number of links</strong></td>
<td>$N + \sqrt{N}$</td>
</tr>
<tr>
<td><strong>Diameter</strong></td>
<td>$\frac{3}{2} \sqrt{N}$</td>
</tr>
<tr>
<td><strong>Bisection bandwidth</strong></td>
<td>2</td>
</tr>
</tbody>
</table>

Each of the $\sqrt{N}$ local rings contains $\sqrt{N}$ links. The single global ring contains $\sqrt{N}$ links.

The diameter within a ring is $\frac{\sqrt{N}}{2}$. In the worst case, such a diameter is crossed for three rings: one end of a local ring to the bridge to enter the global ring, to the most distant bridge to exit the global ring, then to the end of that local ring.
**Bonus Question (10 points)**

Warning: This question is harder than others, so we recommend finishing as much of the quiz as you can before attempting it!

Derive the average distance (in hops or links) for a Hierarchical Ring topology as a function of the number of nodes in the network, \( N \). Assume uniform random traffic (where each node sends \( 1/N \)th of the traffic to each destination, including itself). To ease your derivations, you can define a variable \( k = \sqrt{N} \), and assume \( k \) is an even integer. *Only exact answers are accepted, no partial credit.*

With probability \( \frac{1}{k} \) the source and destination are in the same ring.
The average distance in this case is \( \frac{k}{4} \).

With probability \( \frac{k-1}{k} \) the source and destination are in different rings.
The average distance to enter the global ring is \( \frac{k}{4} \) (this includes when the source is a bridge node).
The average distance to exit the global ring is \( \frac{k}{4} \) (this includes when the destination is a bridge).
The average distance to traverse the global ring is *not* \( \frac{k}{4} \), because we have assumed the source and destination are in different rings; we scale the average distance for a ring by \( \frac{k}{k-1} \).

The average distance is therefore:

\[
\frac{1}{k} \cdot \frac{k}{4} + \frac{k-1}{k} \left( \frac{k}{4} + \frac{k}{4} + \frac{k}{k-1} \right) = \frac{3\sqrt{N} - 1}{4}
\]