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Veitch's Theorem

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This note presents a general model for an asynchronous element, and in terms of it presents a rewriting of the proof of a theorem by E. A. Veitch (unpublished paper 1964), to the effect that it is impossible to pass signals down an asynchronous line without feedback. References are to my report, "Badly timed elements and well timed nets." (Report # 65-02, Moore School of Electrical Engineering, University of Pennsylvania, June 1964.)

Roughly speaking, an asynchronous element, as conceived here, is described by a state graph in which transitions take an amount of time that is subject to a time dispersion. But another important conceptual aspect concerns how long an input signal must be present to cause a transition.

An asynchronous element has a finite number of states, $S_1, \ldots S_q$, a number of element inputs each capable of a finite number of values, and a number of element outputs each capable of a finite number of values. The element at any time may be in one of the states or in transition from one state to another. The value of each element output depends only on the state; during the time that the element is in transition from one state to another an element output takes the value associated with the first state.

A transition will take place if certain element inputs assume certain values for a sufficiently long time during which the element is in a certain state and not already in transition. More precisely, for each S_i and S_j, $i \neq j$, there is a condition C_{ij} on the inputs such that if the element is in

S_i and the input condition C_{ij} obtains simultaneously for a certain length of time x (rise time) them the element will make a transition from S_i to S_j. There is no way of predicting what the rise time x will be except that it will be between the limits $\lambda(1-\epsilon)$ and $\lambda(1+\epsilon)$. For $j \neq j'$, C_{ij} and $C_{ij'}$ are mutually exclusive. C_{ij} may be impossible, in which case the element can never make a direct transition from S_i to S_j. Or it may be universal, in which case the transition occurs whenever the element is in S_i. If C_{ii} obtains while the element is in S_i, it simply remains in S_i and does not make any transition at all. $\bigcup_{j=1}^{q} C_{ij}$ must be universal.

For $i \neq j$, the amount of time between the beginning of the time when both the element is in S_i and the condition C_{ij} obtains and the time when the element begins to be in state S_j is y (reaction time); y is unpredictable except that $(1-\epsilon)\tau \leq y \leq (1+\epsilon)\tau$. The constants ϵ, λ, τ must satisfy the condition $(1-\epsilon)\tau > (1+\epsilon)\lambda$; for otherwise a reaction might be completed before the end of the rise time.

An asynchronous element as formulated is oblivious to its input condition while it is in transition from one state to another. This feature is not necessary, but insures that the element be precisely and realistically formulated. If there is an input condition that has one effect when the element is in S_i and a contrary effect when the element is in S_j then it seems just as well to assume that that input condition has no effect while the element is in transition from S_i to S_j. On the other hand, there are circumstances under which it does not seem to be undesirable to allow the element to be influenced by its input condition while in transition. The RBF "and" and "or" gates are a departure from the above formulation since they do allow an input condition to influence the element in transition. For example, the "and" gate may begin a transition to Q_o in the midst of a transition from P to Q_d. It is clear that this feature of the RBF "and" and "or" gates is harmless.

It is to be noted that although an asynchronous element in its generality has any number of states, useful examples will have a small number of states, say less than ten. More complicated functions should be achieved by constructing nets by cascading elements. The asynchronous elements of the Report were all reasonably small. But the interesting thing about Veitch's theorem is that it holds for a chain of asynchronous elements, however complex the elements.

As a concluding observation, let us note that computing elements have been divided into fixed and growing, into discrete and continuous, into synchronous and asynchronous, and into deterministic and probabilistic. It is worthwhile to note that an asynchronous element as formulated above is asynchronous, but it is also fixed, discrete and deterministic. It is fixed because it does not grow in time. It is discrete because there is a finite discrete set of input and output values and states. And it is deterministic because an input condition of sufficient duration will have a determined effect, even though the timing is not precisely predictable. It would appear that the element as formulated above is general enough to embrace all possible asynchronous, fixed, discrete, deterministic elements, and therefore is suitable for the most general of questions, such as that answered by Veitch's theorem. But it is worth noting that corresponding questions about other types of elements might prove to be equally interesting. For example, is there a continous asynchronous delay element, an infinite chain of which without feedback can pass signals at a constant repetition rate?

Given the precise notion of an asynchronous element, Veitch's theorem can be stated and proved. An <u>infinite uniform chain of elements without feedback</u> is an infinite sequence of elements E_1, E_2, . . ., all alike and such that, for each i, the element outputs of E_i are connected directly only to the element inputs of E_{i+1}. Such a chain passes successfully an infinite sequence of signals if when the sequence is, in some coded form or other, placed at the inway (the element input of the first element) it is possible to deduce the entire sequence from the history of any element (i.e. the succession of states) and the knowledge of how the sequence was coded at the inway.

Veitch's Theorem. An infinite uniform chain of asynchronous elements without feedback cannot in the worst case pass successfully an infinite sequence of signals that are placed at the inway in any coded form at a uniform repetition rate.

It seems plausible that this theorem would still be true if the last phrase "at a uniform repetition rate" were deleted. If so, it is certainly not obvious. The difficulty comes in cases where the timing of the signals at the inway is such that longer and longer intervals ensue between successive signals; for example, where the r^{th} signal occurs at the inway at time 2^r.

It turns out that the word "uniform" can be deleted, and the theorem is still true. For a slight modification in Lemma 1 is all that is needed to generalize the proof. The surprising thing here is that the theorem is true even for those chains whose elements are capable of an unbounded amount of storage, as long as each element is capable of only a finite amount of storage. Needless to say, the theorem would be false if individual elements were allowed to have unbounded storage.

Before proceeding with the rigorous proof let us consider the matter intuitively. Veitch's theorem is a generalization of the proof that the infinite chain of neural delays does not pass more than a certain number of signals successfuly. It will be recalled that that proof procedes by establishing that an arbitrarily long burst of signals can appear at a sufficiently advanced point in the chain at a repetition rate that is too fast for a slowly reacting element. One difficulty in generalizing is that a general asynchronous element is capable of storing up to a certain number, k, of signals whereas the neural delay could store at most one. (In a certain sense, the neural delay does not store at all, since it releases its signal immediately after absorbing it.) Another difficulty is that in an infinite chain of some type of asynchronous element it might be possible by some feed-forward principle to prepare the way for a signal by sending ahead a warning signal: "Get rid of one of the signals that you have because there is another signal on its way." Finally, the proof is made more difficult by the generality of the formulation of the asynchronous element.

The proof is by reductio ad absurdum. It is assumed that there is a repetition rate at the inway such that for any set of timing reactions all of the infinite sequence of signals will be passed successfully. Then it is shown that whatever repetition rate signals may appear at any point, it is possible for signals to appear at another point at a significantly faster rate. The assumptions are thus shown to imply that when an infinite chain of elements processes an infinite set of signals there is a set of timing reactions such that there will be no bound on the repetition rate of signals at points within the chain, which contradicts the assumption that the signals are passed successfully.

The storage restriction criterion, which is the principle that an element cannot hold information about more than a constant k signals at any time, implies that if a long burst of signals appears at a point in the chain at a certain rate then at another point they must appear, whatever the timing reactions of the elements between, at a rate which is approximately the same. This must be true, if the elements are to process the signals successfully, even when the elements react as slowly as possible. But this implies that, when the timing reactions of the elements are as fast as possible, the rate will be significantly faster at the second point. An important part of this train of thought is that, since there is no feedback, elements react independently of other elements that are ahead in the chain. Also, once signals are safely past a certain point, what happens to them is independent of the actions of elements before that point.

However, the above is a gross oversimplification of the proof. The difficulties that arise in making this train of thought rigorous are enough to necessitate the lengthy and detailed procedure that follows.

Let $T(r,n)$ be the ealiest time that the value of the r^{th} signal (zero or one) can be derived from the history of E_n. Thus E_n arrives at a certain state at time $T(r, n)$ and from the history of the sequence of states that E_n assumes up to and including $T(r, n)$, the value of the r^{th} signal can be inferred. The following assertions ((1) through (4)) are easily proved or otherwise justifiable.

(1) $T(r+1, n) \geq T(r, n)$. Although it is possible that the $(r+1)^{th}$ signal goes through before the r^{th} signal, it is clear that there is no advantage in it. There is, therefore, no loss of genrality in assuming (1).

(2) $T(r, n+1) \geq T(r, n) + \delta$, where δ is the minimal propagation time of a signal from element to element. Certainly, $\delta \geq (1-\epsilon)\eta > 0$. The values $T(i, j)$ depend on the input-signal reception at the inway and the various timing reactions at the elements (which are rise times and transition times). Since there is no feedback,

(3) $T(i, j)$ does not depend on any timing reaction of E_n for $n > j$.

Let $k = \log_2 q$, where q is the number of states of each element. Then at any time the element may contain information about at most k signals. At time $T(r+k, n)$, therefore, and for ever after, E_n can have no information about the r^{th} signal which must have arrived at E_{n+1}. Thus

(4) $T(r, n+1) \leq T(r+k, n)$

Let $RI(r, r+s, n) = \frac{1}{s}(T(r+s, n) - T(r,n))$. Note that if $T(r,n)$ and $T(r+m, n)$ are identified, respectively, with the arrival of the r^{th} and $(r+m)^{th}$ signals at E_n then $RI(r, r+s, n)$ is the average repetition interval at ~~X_n~~ E_n between the r^{th} and $(r+s)^{th}$ signals.

Veitch's theorem follows easily from the theorem below. Let A be the set of all β such that, for every s, there are a possible set of timing reactions an $\underline{r}$ and and $\underline{n}$ such that $RI(r, r+s, n) \leq \beta$.

Theorem. Under the assumption that the infinite chain passes all signals successfully, if $\beta \in A$ then $\beta(1-\epsilon) \in A$.

Lemma 1. For a given β, m, s, there is an $s' \geq s$ such that for any n and r, if $RI(r, r+s'+mk, n) \leq \beta$ then, for any possible set of timing reactions, $RI(r, r+s', n+m) \leq \beta(1+\epsilon)$. (Note that s' does not depend on n or r).

Proof: By (4), $T(r+s', n+m) \leq T(r+s'+mk, n)$, for whatever s' we choose. Furthermore, $T(r, n+m) \geq T(r, n) + m\delta$, by (2). Hence, $T(r+s', n+m) - T(r, n+m)$ $\leq T(r+s'+mk, n) - T(r, n) - m\delta$. If we assume that $T(r+s'+mk, n) - T(r, n) = (s'+mk)\beta$ we then get $T(r+s', n+m) - T(r, n+m) \leq (s'+mk)\beta - m\delta = s'\beta + (mk\beta - m\delta)$. Hence $RI(r,r+s', n+m) \leq \beta + \frac{mk\beta - m\delta}{s'}$. By taking s' sufficiently large, the quanity $\frac{mk\beta - m\delta}{s'}$ can be made as small as we please; thus we can make $RI(r, r+s', n+m) \leq \beta(1+\epsilon)$, which must be true regardless of timing. The choice of s' depends on m and β as well as the constants k and δ, but not on n or r.

Intuitively, Lemma 1 states that, regardless of reaction timing, the repetition interval of signals at E_{n+m} cannot be too much more than the spacing of signals at E_n; otherwise the elements would have to store an arbitrarily large number of signals. In particular, this means that even under the set of slowest reaction timings of the elements $E_{n+1}, \ldots, E_{n+m}$ the repetition interval of signals at E_{n+m} must be no more than $(1+\epsilon)$ times the repetition interval at E_n. Lemma 3 goes on to show, roughly speaking, that under the fastest set of reaction timings the repetition interval at E_{n+m} must be no more than $1-\epsilon$.

Lemma 2. Under the same hypothesis of Lemma 1, and given the s' satisfying it, for some i, $0 \leqq i \leqq s' - s$, $RI(r+i, r+i+s, n+m) \leqq \beta(1+\epsilon)$.

Lemma 1 says that the average repetition interval of the s' signals is $\leqq \beta(1+\epsilon)$. Lemma 2 says that, for some subset of s consecutive signals of the set of s', the repetition interval is $\leqq \beta(1+\epsilon)$. The proof is elementary, given that Lemma 1 has been established.

Lemma 3. Suppose that, for a certain fixed set of timing reactions of all elements before time t, and whatever the timing reactions of the elements after time t, $T(r+s, n+m) - m\delta < t \leqq T(r, n+m)$; and suppose that $RI(r, r+s, n+m) = \rho_1$ under the assumption that all timing reactions of $E_{n+1}, \ldots, E_{n+m}$ after time t are fast, and $= \rho_2$ under the assumption that they are all slow. Then

$$\frac{\rho_1}{\rho_2} = \frac{1-\epsilon}{1+\epsilon}.$$

(Reactions that begin before t and end after t are changed according to the proportion of time after t. Thus if a reaction begins at t-x and ends at $t+y(1-\epsilon)$ in the first alternative it will begin at t-x and end at $t+y(1+\epsilon)$ in the second.)

Proof: Since $T(r+s, n+m) - m\delta < t$, what happens at E_n after t cannot have any influence on $T(r+s, n+m)$, since $m\delta$ is the shortest possible propagation time from E_n to E_{n+m}. Thus the reactions of $E_1, \ldots, E_n$ after time t are immaterial to $T(r+s-i, n+m)$ for any $i \geqq 0$. Only the reactions of $E_{n+1}, \ldots, E_{n+m}$ need be considered after time t. Furthermore, given the fixed set of timing reactions and given the timing reactions after time t of $E_{n+1}, \ldots, E_{n+m}$, $T(r, n+m)$ and $T(r+s, n+m)$ are determined.

By definition

Now $RI(r, r+s, n+m) = \frac{1}{s} \{T(r+s, n+m) - T(r, n+m)\}$; and $T(r+s, n+m) \geqq T(r, n+m) \geqq t$, by (1) and hypothesis of Lemma 3. Hence, in order to prove Lemma 3, if is sufficient to show that the ratio of each of the quantities $T(r+s, n+m) - t$, and $T(r, n+m) - t$ under the assumption of fast timing reactions of $E_{n+1}, \ldots, E_{n+m}$ after time t to what it is under the assumption of slow timing reactions is $\frac{1-\in}{1+\in}$. But both of these ratios must be $\frac{1-\in}{1+\in}$, since the effect of fast timing is simply to speed every reaction by $\frac{1-\in}{1+\in}$.

A reaction will take place under the assumption of fast reactions if and only if it takes place under the assumption of slow reactions, because the reaction thresholds are also subject to the same speedup. The effect of the speedup of the elements $E_{n+1}, \ldots, E_{n+m}$ after time t is simply to compress the time scale of everything that happens by $\frac{1-\in}{1+\in}$.

We now complete the proof of the theorem. Assume that $\beta \in A$. In order to prove $\beta(1-\in) \in A$, it must be proved that, for an arbitrary choice of s, there are a possible set of timing reactions, an r, and an n such that $RI(r, r+s, n) \leqq \beta(1-\in)$. By hypothesis, there are r^0, n^0 and timing reactions such that $RI(r^0, r^0+s^0, n^0) \leqq \beta$, for every s^0.

Given β and s, let m be determined as follows. Assuming that $RI(r+i, r+i+s, n+m) \leqq \beta(1+\in)$ in any possible set of timing reactions, m is taken to be sufficiently large so that $T(r+i+s, n+m) - m\delta < T(r+i, n+m)$ in any possible set of timing reactions. That there is such a value of m, namely one satisfying $m > \frac{1}{\delta}(T(r+i+s, n+m) - T(r+i, n+m))$, follows from the fact that $T(r+i+s, n+m) - T(r+i, n+m) = sRI(r+i, r+i+s, n+m) \leqq s\beta(1+\in)$. The value of m depends on δ, s, β and $\in$. Having so selected m, it will be possbile below to apply Lemma 3.

Please revise the two meanings of "$\in$"

Having determined m, we can now apply Lemma 2. Thus there is an s' such that (for any n and r) if RI(r, r+s'+mk, n) $\leqq \beta$ then for some i, $0 \leqq i \leqq s'-s$, RI(r+i, r+i+s, n+m) $\leqq \beta(1+\epsilon)$, for any possible set of timing reactions. But, by hypothesis there are n and r and a possible timing of the elements $E_1, \ldots, E_n$ such that RI(r, r+s'+mk, n) $\leqq \beta$. For this set of timing reactions of $E_1, \ldots, E_n$ and for timing reactions of $E_{n+1}, \ldots, E_{n+m}$ that are as slow as possible, it follows that RI(r+i, r+i+s, n+m) $\leqq \beta(1+\epsilon)$. Since m has already been taken large enough so that T(r+i+s, n+m) - mδ < T(r+i, n+m) a value between these can be chosen for the t of Lemma 3 (substituting r+i for r). From Lemma 3 we infer that for the set of reaction timings that are the same as before except that the reactions of $E_{n+1}, \ldots, E_{n+m}$ after t are as fast as possible, RI(r+i, r+i+s, n+m) $\leqq \beta(1-\epsilon)$. This concludes the proof of the theorem.