LABORATORY FOR COMPUTER SCIENCE

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

The Recursive Equivalence of the Reachability Problem and the Liveliness Problem for Petri Nets and Vector Addition Systems

Computation Structures Group Memo 107

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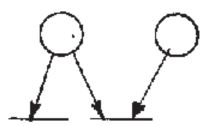
The following are a few notes about a class of Petri Nets that are very similar to Free Choice Nets and strictly include all Well-Formed Free Choice Nets (WFFC Nets). Refer to MAC-TR-94 for terminology and notation details.

A. Recapitulation

In a Free Choice net, no arc goes from a shared place to a shared transition:

(FC)



does not occur in a Free Choice Net.

Figure 1

In formal notation*: $p.t \Rightarrow (p^{\bullet} = \{t\}$ <u>or</u> ${}^{\bullet}t = \{p\})$.
This guarantees that when a token arrives in p, <u>all</u> transitions in $p^{\bullet}$ are equally enabled and in conflict, thus a <u>free choice</u> can be made as to which $t \in p^{\bullet}$ to fire next. It should be noted that a sufficient condition for this is:

(EFC)

$$\forall p,q \text{ places: } p^{\bullet} \cap q^{\bullet} \neq \emptyset \Rightarrow p^{\bullet} = q^{\bullet}$$

We call this the <u>Extended</u> Free Choice condition. Fred Commoner calls this simply Free Choice .

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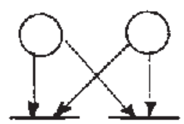
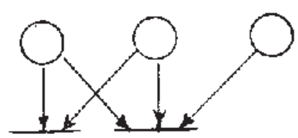
*MAC-TR-94, p. 42.

Example:



is Extended Free Choice (EFC)

is not EFC



is not EFC

Figure 2

All theorems and statements about FC nets in MAC-TR-94 apply to EFC, except that Allocations have to be redefined; see the remark on p. 3.

One reason is that the property of FC nets used in proofs is actually the EFC-property. Another reason is the simple equivalence:

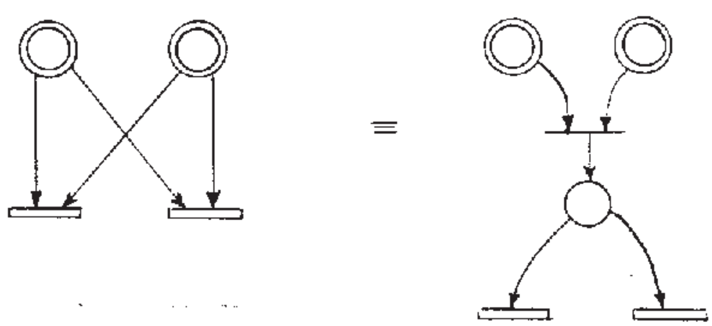


Figure 3

The distinction is usually not important, except maybe in the context of Fred Furtek's thesis.

(WF) An FC net is said to be Well-Formed (WF) iff it has a live-and-safe (LS) marking.

From the Live-and-Safeness Theorem of MAC-TR-94 follows*:

> An FC net has an LS marking iff it is covered by Strongly Connected State Machines (SCSM's) and every minimal deadlock is an SCSM.

(SM-allocation) A <u>State-Machine</u> <u>Allocation</u>** over an FC net $\langle \Pi, \Sigma \rangle$ is a function $B: \Sigma \rightarrow \Pi$ such that:

$$(\forall t \in \Sigma) \quad B(t) \in {}^{\bullet}t$$

<u>Remark</u>: If we use the EFC definition, the allocation must satisfy the following consistency requirement:

$$\forall t_1, t_2: \quad {}^{\bullet}t_1 \cap {}^{\bullet}t_2 \neq \emptyset \quad \Rightarrow \quad B(t_1) = B(t_2)$$

This is always possible since ${}^{\bullet}t_1 = {}^{\bullet}t_2$, which follows from the reverse-dual of the EFC-condition:

$${}^{\bullet}t_1 \cap {}^{\bullet}t_2 \neq \emptyset \quad \Rightarrow \quad {}^{\bullet}t_1 = {}^{\bullet}t_2$$

It can easily be shown that this reverse-dual† condition is equivalent to $p_1^{\bullet} \cap p_2^{\bullet} \neq \emptyset \quad \Rightarrow \quad p_1^{\bullet} = p_2^{\bullet}$.

(SM-reduction) Given an SM-allocation $B: \Sigma \rightarrow \Pi$, we can <u>reduce</u> an FC net as follows‡:

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*MAC Technical Report TR 94, p. 63

**ibid., p. 71

†ibid., p. 67

‡ibid., p. 71

Step 1: Delete all unallocated places. $(\Pi - B(\Sigma))$.

Step 2: Delete all transitions that have all output places already deleted.

Step 3: Delete all places that have at least one output transition already deleted.

Repeat Steps 2 and 3 until neither is applicable anymore.

What is left over is an SM-reduced net. A given net usually has several distinct SM-reduced nets. In general, a reduced net may be empty (the algorithm has deleted all places and all transitions), it may be a collection of one or more SCSM's, or it may be something else.*

The Well-Formedness Theorem** says, among other things:

> An FC net is WF iff every SM-reduction is a collection of one or more SCSM's.

For the sake of completeness, it may be mentioned that we also define Marked-Graph allocations and reductions, in a reverse-dual way. See MAC TR-94 for details.

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* ibid. p. 73

**ibid. p. 81

B. State-Machine-Allocatable Nets

In this section we consider what happens if we apply the SM-reduction algorithm to an arbitrary Petri Net. We define an SM-allocation over a Petri Net $\langle \Pi, \Sigma \rangle$ as before as a function:

$$B: \Sigma \rightarrow \Pi$$

such that:

$$(\forall t \in \Sigma) \quad B(t) \in {}^{\cdot}t$$

We do <u>not</u> add the condition of page 3 to make it compatible with EFC, because if a net is not EFC, yet contains two transitions t_1 and t_2 such that ${}^{\cdot}t_1 \subseteq {}^{\cdot}t_2$, there would be no allocation whose range includes any place in ${}^{\cdot}t_2 - {}^{\cdot}t_1$. Thus, a certain portion of the net would be missing in every reduced net, and we shall see later that this is undesirable; see Section D on p. 16.

Now we define:

<u>Def</u>:	A Petri Net is SMA iff every SM-reduction is a collection of one or more SCSM's.

Thus, an FC net has an LS marking iff it is SMA.

We shall now show that this theorem is not restricted to FC nets, but that every SMA Petri Net has an LS marking. We shall also see that most results about Net decompositions in Chapter 5 of MAC TR-94 apply to Petri Nets in general, and all apply (some vacuously) to SMA nets.

First, we define the <u>released</u> form of a Petri Net $\langle \Pi, \Sigma \rangle$ where $\Pi = \{p_1, \ldots, p_m\}$ and $\Sigma = \{t_1, \ldots, t_n\}$.

By a shared place we mean a place with several output transitions.

By a shared transition we mean a transition with several input places.

Def: An arc from a shared place p_i to a shared transition t_j becomes released if we modify the net in the following way:

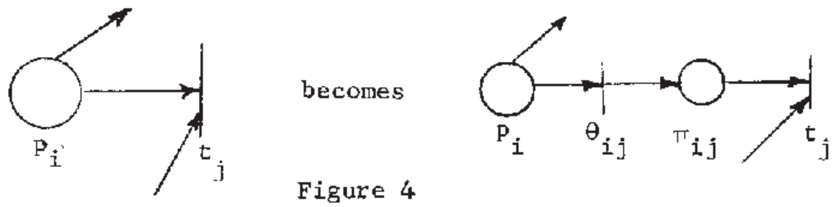

Figure 4

We add a place, labeled π_{ij}, and a transition, labeled θ_{ij}, between p_i and t_j and adjust the $\cdot$ relation:

before: $p_i \cdot t_j$

after: $p_i \cdot \theta_{ij}$, $\theta_{ij} \cdot \pi_{ij}$, $\pi_{ij} \cdot t_j$

$$\text{new } \Pi = \text{old } \Pi \cup \{\pi_{ij}\}$$

$$\text{new } \Sigma = \text{old } \Sigma \cup \{\theta_{ij}\}$$

We say released because there is no longer any constraint as to which way a token in p_i will go.

We could of course "release" an arc from an unshared place or to an unshared transition, but no constraint is removed, and it is easy to see that the firing sequences would not be changed, except for the occasional appearance of a θ label in the sequence. (If we erased the θ's the set of firing sequences would be the same; more about this later, p.11.)

Def: A Petri Net is said to be in released form iff every arc from a shared place to a shared transition has been released, and the marking is unchanged in the original places and zero in the new places.

Example 1

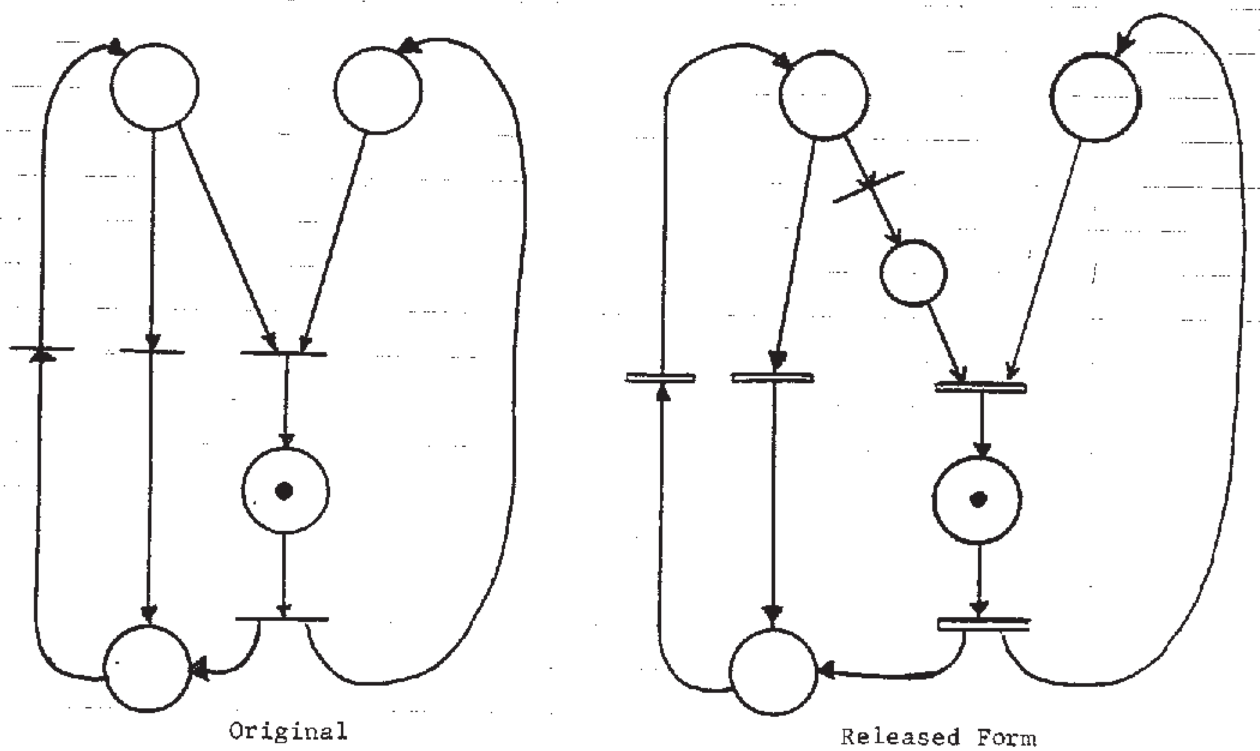

Figure 5

Example 2

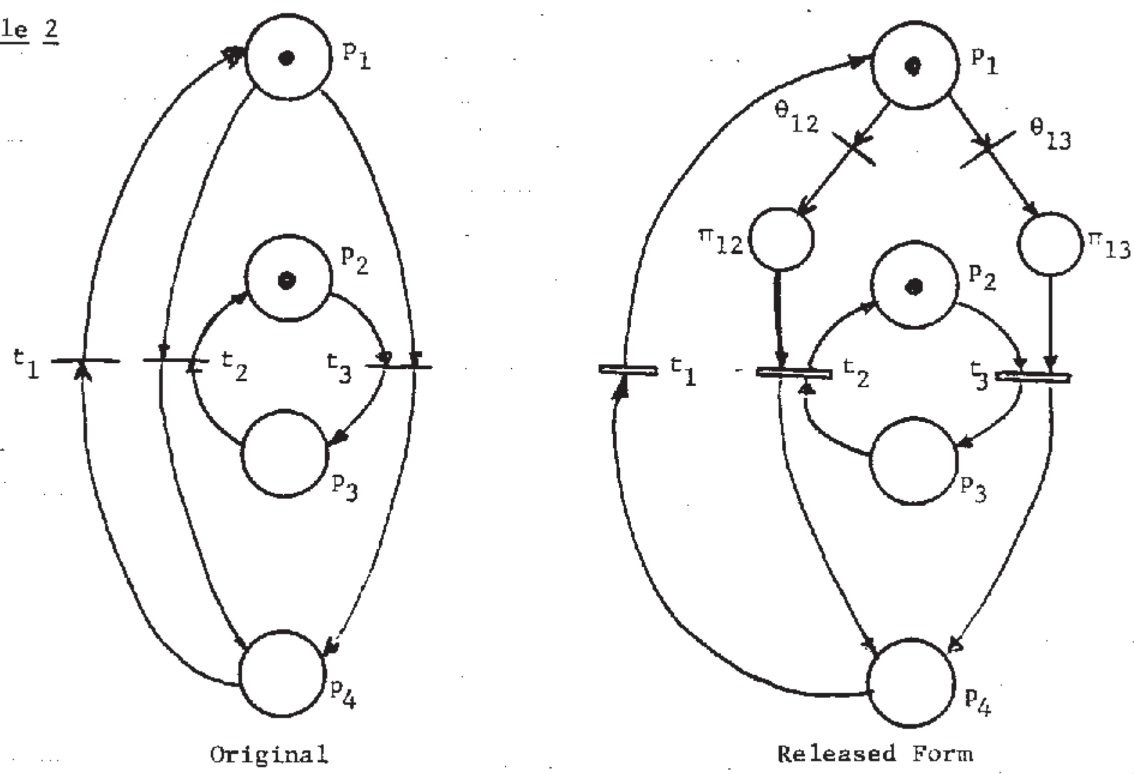

Figure 6

It is clear that:

The released form of a Petri Net is Free Choice, by construction.

It may be remarked that in both examples, the original net is a live, safe, non-FC net. But in example 1 the released form is also live and safe, whereas in example 2, the released form is not live, and actually has no live marking.

<u>Theorem 1</u>: a) If B is an SM-allocation over a Petri Net $Q = \langle \Pi, \Sigma \rangle$ yielding an SM-reduced net Q_B, there exists a unique SM-allocation B' over the released net $Q' = \langle \Pi \cup \Pi', \Sigma \cup \Sigma' \rangle$ such that the SM-reduced net $Q'_{B'}$ contains the same p-labeled places and the same t-labeled transitions as Q_B, and is homeomorphic* to it.

b) Moreover, every SM-allocation of Q' corresponds in this way to a unique SM-allocation of Q.

Notation: $\Pi = \{p_i \ldots\}, \quad \Sigma = \{t_j \ldots\}$

$\Pi' = \{\pi_{ij} \ldots\}$ and $\Sigma' = \{\theta_{ij} \ldots\}$

<u>Proof</u>: a) Define B' as follows:

$$B(t_j) = p_i \quad \Leftrightarrow \quad \begin{cases} B'(t_j) = \pi_{ij} \ \ \& \ \ B'(\theta_{ij}) = p_i & \text{(if } p_i \cdot t_j \text{ is released)} \\ B'(t_j) = B(t_j) & \text{(if } p_i \cdot t_j \text{ not released)} \end{cases}$$

Now, B' is clearly an SM-allocation, since $\pi_{ij} \in {}^{\cdot}t_j$ and $\{p_i\} = {}^{\cdot}\theta_{ij}$ by construction. Let us follow the reduction algorithm.

B deletes: ⇔	B' deletes	Condition:
p_i in step 1	p_i in step 1	$p_i \cdot t_j$ $p_i \neq B(t_j)$ (unreleased)
p_i in step 1	π_{ij} in step 1 θ_{ij} in step 2 p_i in step 3	$p_i \cdot \theta_{ij} \cdot \pi_{ij} \cdot t_j$ $p_i \neq B(t_j)$ (released)
t_j in step 2	t_j in step 2	(${}^{\cdot}t_j$ same in Q and in Q')

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*homeomorphic = same graph up to vertices of order 2: The π_{ij} and θ_{ij}.

B deletes:	⇔	B' deletes	Condition:
p_i in step 3		p_i in step 3	$p_i \cdot t_j$, where t_j deleted (unreleased)
p_i in step 3		π_{ij} in step 3 θ_{ij} in step 2 p_i in step 3	$p_i \cdot \theta_{ij} \cdot \pi_{ij} \cdot t_j$ where t_j deleted (released)

Thus we see that B deletes p_i <u>iff</u> B' deletes p_i (together with any π_{ij} and θ_{ij} in fact); therefore the reduced nets agree on all p_i, and if the reduced released net contains p_i, it also contains the associated π_{ij}'s and θ_{ij}'s. Also, the reduced nets agree on all t_j.

b) Q and Q' have the same number of SM-allocations: The product of the in-degree of all transitions. The only difference in terms of transitions are one-input θ-transitions which do not contribute to the product.

QED

<u>Corollary 1</u>: The relased form of an SMA-net is a WFFC net.

In fact, the quasi-identity of the SM-reductions of a Petri Net and its released form permit us to extend several results of MAC TR-94, which we list without further proof: (in parentheses, page reference to MAC TR-94)

- The reverse-dual of an SMA net is SMA.
- Every SM-reduced net is a closed subnet defined by a non-decreasing trap.* (73)
- Every MG-reduced net is a conflict-free open subnet.* (77)
- The SM-reductions and MG-reductions of an SMA-net cover the net (80); every MG-reduction is a collection of Strongly Connected Marked Graphs.

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*These two theorems do not even depend on the concept of released arcs. Also for reasons of conflicting terminology with later literature, we do not use "consistent" in the way it was defined on page 56 of MAC TR-94. Thus, we now say "closed subnet" instead of "closed consistent subnet."

C. Behavior of Petri Nets, firing sequence agreement

- Notation: Given a Petri Net $\langle \Pi, \Sigma \rangle$, initial marking M_0. A firing sequence σ can be viewed as a string of transition labels, which we write as: $\sigma \in \Sigma^*$. Of course, in general, not every string is a firing sequence.

If $T \subseteq \Sigma$, we may consider the string σ' obtained from a given string σ by deleting all labels not from T in the string as: $\sigma' = \sigma \cap T$.

Example: abcbdac $\cap \{a, b\}$ = abba

To say that transition t is fired by firing sequence σ, i.e. that label t appears in string σ, we write: $t \in \sigma$. We have:

$$t \in \sigma \Leftrightarrow \sigma \cap \{t\} \neq \lambda \qquad (\lambda \text{ is the empty string})$$

Def: Two firing sequences σ and σ' agree over a subset of transitions $T \subseteq \Sigma$ iff $\sigma \cap T = \sigma' \cap T$. We write this as:

$$\sigma \equiv \sigma' \bmod T \quad \stackrel{\Delta}{\Leftrightarrow} \quad \sigma \cap T = \sigma' \cap T$$

We extend the notion of agreement to sets of sequences S, S':

$$S \equiv S' \bmod T \stackrel{\Delta}{\Leftrightarrow} \left| \begin{array}{l} (\forall \sigma \in S)(\exists \sigma' \in S')\ \sigma \equiv \sigma' \bmod T \\ \& \ (\forall \sigma' \in S')(\exists \sigma \in S)\ \sigma \equiv \sigma' \bmod T \end{array} \right.$$

- More notation about released forms:

Let $Q = \langle \Pi, \Sigma \rangle$ be a Petri Net and let $Q' = \langle \Pi \cup \Pi', \Sigma \cup \Sigma' \rangle$ be its released form. Let the elements of Π be denoted p_i, $1 \leq i \leq |\Pi|$

"	Π'	"	π_{ij}, for some pairs i, j
"	Σ	"	t_j, $1 \leq j \leq |\Sigma|$
"	Σ'	"	θ_{ij}, for some pairs. i, j

where the π_{ij} and θ_{ij} are defined as on page 6.

Let ${}^\cdot t_j = \{p_{i_1}, p_{i_2}, \ldots p_{i_k}\}$ in the original net Q. Then we denote the set of the pre-transitions of t_j in the released net Q' by:

$$\bar{\theta}_j = \{\theta_{i_1 j}, \theta_{i_2 j} \ldots, \theta_{i_k j}\}$$

By $\vec{\theta}_j$ we denote a string which contains each element of $\overline{\theta_j}$ exactly once, for example:

$$\vec{\theta}_j = \theta_{i_1j}\ \theta_{i_2j}\ \dots\ \theta_{i_kj}$$

The order does not matter; and $\vec{\theta}_j$ can of course be the empty string, if in Q no input place to t_j is shared, or if t_j is single-input.

Similarly, let $p_i^{\bullet} = \{t_{j_1}, \dots t_{j_k}\}$ in Q. Then we define:
$\overline{\pi_i} = \{\pi_{ij_2}, \dots \pi_{ij_k}\}$ in Q'.

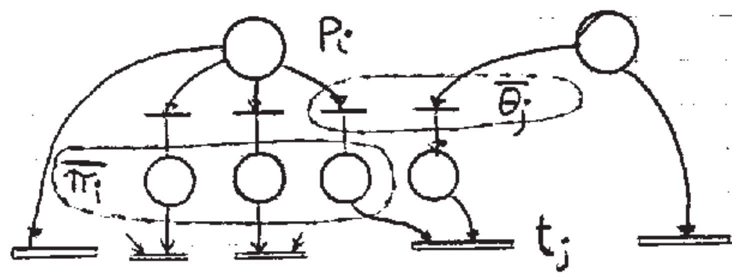

Figure 7

Theorem 2: Let Q be a Petri Net and Q' its released form (see notation above). Let σ be a firing sequence of Q and σ' be a firing sequence of Q' such that σ and σ' agree on the transitions of Q. Then the markings of Q and Q' reached by σ and σ' correspond to each other in the following way:

$$\sigma = \sigma' \cap \Sigma : \quad (\forall i) \quad (M_0[\sigma\rangle)(p_i) = (M_0'[\sigma'\rangle)(\{p_i\} \cup \overline{\pi_i})$$

The number of tokens in a place of the original net is equal to the number of tokens on that place in the released net plus its associated π-places.

Proof:

- At the initial marking, we have:

$$\forall i: \quad M_0(p_i) = M_0'(\{p_i\} \cup \overline{\pi}_i)$$

- Let M and M' be markings of Q and Q' respectively such that:

$$\forall i: \quad M(p_i) = M'(\{p_i\} \cup \overline{\pi}_i)$$

- Let $\sigma'' \in \Sigma'^*$ be a sequence of θ-firings only.

Clearly:

$$\forall i: \quad (M'[\sigma''\rangle)(\{p_i\} \cup \overline{\pi}_i) = M'(\{p_i\} \cup \overline{\pi}_i)$$

- Consider a t-firing: Then we have:

$$\forall i: \quad (M[t_j\rangle)(p_i) = (M'[t_j\rangle)(\{p_i\} \cup \bar{\pi}_i)$$

Thus, by induction we can say that:

$$\sigma = \sigma' \cap \Sigma \Rightarrow \forall i (M_0[\sigma\rangle)(p_i) = (M_0'[\sigma'\rangle)(\{p_i\} \cup \bar{\pi}_i)$$

QED

Theorem 3: Let $Q = \langle \Pi, \Sigma \rangle$ be a Petri Net and $Q' = \langle \Pi \cup \Pi', \Sigma \cup \Sigma' \rangle$ be its released form. Then the firing sequences of the original net agree over Σ with those of the released form:

$$\left.\begin{array}{l} S = \{\sigma \mid \text{firing sequence of } Q\} \\ S' = \{\sigma' \mid \text{firing sequence of } Q'\} \end{array}\right) \quad S \equiv S' \bmod \Sigma$$

Proof: a. If σ is a firing sequence of Q, let σ' be the firing sequence obtained by replacing each occurrence of t_j by the string $\vec{\theta}_j t_j$. Then σ' can easily be seen, by induction, to be a firing sequence of Q':

Basis: At the original markings M_0 for Q and the corresponding M_0' for Q', we have: (by definition)

$$M_0'/\Pi = M_0 \qquad (M_0' \text{ restricted to original places})$$

$$M_0'(\Pi') = 0$$

Inductive Step: Now assume a marking M for Q and a corresponding M' in Q' which agrees with M on Π and is blank on Π':

$$M'/\Pi = M \quad \& \quad M'(\Pi') = 0$$

If t_j is firable at M in Q, then the set $\overline{\theta_j}$ is firable at M' in Q', i.e. $\vec{\theta}_j$ is firable at M'. Then, at $M'[\vec{\theta}_j\rangle$, t_j is firable in Q', which leads to $M'[\vec{\theta}_j t_j\rangle$:

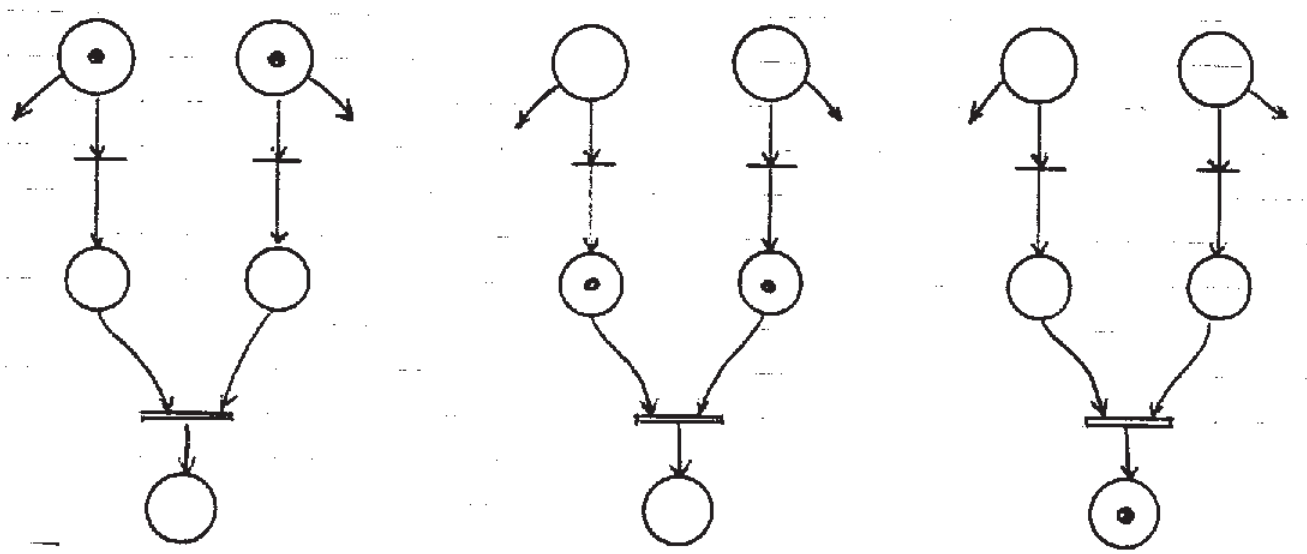

Figure 8

Also, $M'[\vec{\theta}_j t_j\rangle/\Pi = M[t_j\rangle$ and $(M'[\vec{\theta}_j t_j\rangle)(\Pi') = 0$

Thus, every firing sequence σ of Q agrees with some firing sequence σ' of Q'.

b. Now, assume there exists a firing sequence σ' of Q' such that no firing sequence of Q agrees with it on Σ, and let σ' be the <u>shortest</u> such sequence:

$$\sigma' \in S' \text{ such that: } \sigma' = \sigma_1' t_j$$

and σ_1' agrees with some $\sigma_1 \in S$, but t_j is not firable in Q at marking $M_0[\sigma_1\rangle$. We shall prove that this cannot be so.

From Theorem 2, we can now assert that, since $\sigma_1 = \sigma_1' \cap \Sigma$:

$$(\forall i)\ (M_0[\sigma_1\rangle)(p_i) = (M_0'[\sigma_1'\rangle)(\{p_i\} \cup \overline{\pi}_i)$$

Now, since t_j is not firable at $M_0[\sigma_1\rangle$ in Q:

$$\exists k:\ p_k \in {}^{\cdot}t_j \quad \& \quad (M_0[\sigma_1\rangle)(p_k) = 0$$

Therefore:

$$(M_0'[\sigma_1'\rangle)(\{p_k\} \cup \overline{\pi}_k) = 0$$

But this contradicts the assumption that t_j is firable in Q' at $M_0'[\sigma_1'\rangle$, since this implies $(M_0'[\sigma_1'\rangle)(\pi_{kj}) > 0$, and $\pi_{kj} \in \bar{\pi}_k$.

This completes the proof of Theorem 3.

QED

<u>Lemma 1</u>: If the released form Q' of a Petri Net Q is live, then the original net is also live.

<u>Proof</u>: Suppose the original net is not live: There exists a firing sequence σ and a transition t_k such that, after σ, t_k cannot be fired again. Let σ' be the firing sequence of Q' that is obtained from σ by replacing each t_j by $\vec{\theta}_j t_j$ (see p. 12). Then the marking $M_0'[\sigma'\rangle$ of Q' is such that it can be the initial marking of the released form of the net Q with new initial marking $M_0[\sigma\rangle$. Thus the set of firing sequences following σ and σ' in Q respectively Q' still agree over Σ. Since Q' is live, t_k can be fired by some firing sequence, and there is a corresponding firing sequence in Q which must also fire t_k, contrary to our assumption.

QED

Actually, the lemma can be made more specific: If a given t-transition is live in Q', then it is live in Q.

<u>Theorem 4</u>: If the released form Q' of a Petri Net Q has a live marking (not necessarily corresponding to an initial marking of Q according to the definition), then the original net has a live marking.

<u>Proof</u>: In view of Lemma 1, it is enough to show that there must exist a live marking M' of Q' which is a suitable initial marking, i.e. such that $M'(\Pi') = 0$. Now, since Q' is Free Choice, the existence of a live marking implies that every deadlock contains a trap, and every marking which marks a trap in each deadlock is live. Now, if p_i is in some trap T, then we must have $\bar{\pi}_i \subseteq T$. If this were not so, i.e. if we had $p_i \in T$ and $\pi_{ij} \notin T$, then we would have $\theta_{ij} \in T^{\cdot} - {}^{\cdot}T$ (θ_{ij} is a single-output transition). But this contradicts our assumption that T is a trap: $T^{\cdot} \subseteq {}^{\cdot}T$. Therefore, to mark

all traps it is sufficient to mark p-places: There exists a live marking M' which marks Q', the released form of Q under the initial marking M'/Π.

QED

Theorem 5: A Petri Net is safe (bounded) if and only if its released form is safe (bounded).

Proof: This is a direct consequence of Theorems 2 and 3.

Theorem 6: Every SMA net has a live and safe marking:

$$SMA \Rightarrow LS$$

Proof: The released form of an SMA net is a WFFC net (Corollary 1, p. 9) and thus has an LS marking. For the same reason as in the proof of Theorem 4 (p. 14), there is such an LS marking that only marks p-places. Then it follows from Theorems 4 and 5 that the original net has an LS marking.

QED

Theorem 7: A live bounded Petri Net Q has a live released form Q' if and only if it is SMA.

Proof: a. If the released form is live but not WF it must be unbounded.* But then, by Theorem 5, the original net could not be bounded.

b. Assume the net is SMA but the released form Q' is not live. Then there exists a deadlock that contains a blank trap. Since Q' is WFFC that blank trap is in fact a State Machine and therefore ever blank. Now, because of Theorems 1 and 2 there is a corresponding blank SM in the original net Q, which contradicts the assumption that Q is live (any Petri Net is killed by a blank SM, assuming no isolated places).

QED

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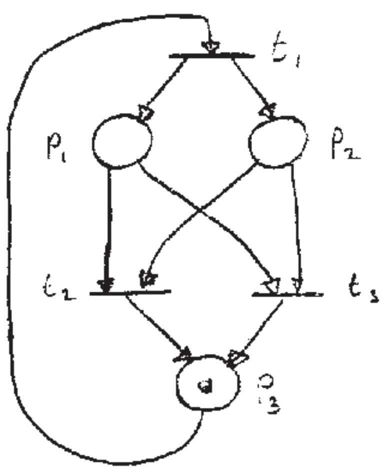
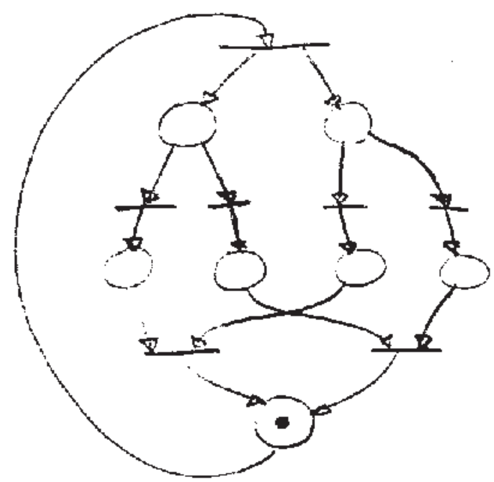
*Lemma 5 in MAC TR-94, p. 59.

D. Clique Reduction and EFC-Nets; Extended SMA Nets

A direct consequence of our decision to use the unmodified definition of SM-allocation (see p. 5) is that the following EFC net is not SMA, and its released form is not live:



EFC net, not FC



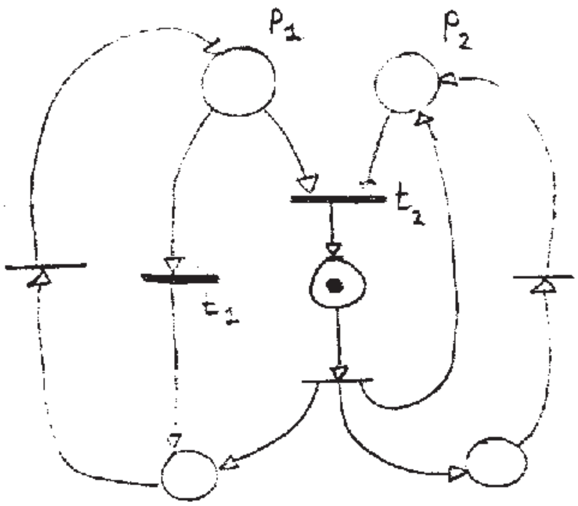
Corresponding released form

Figure 9

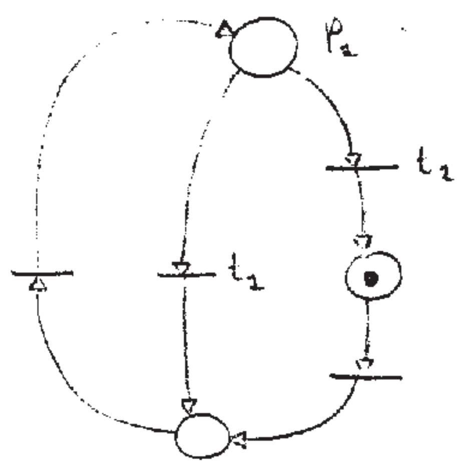
On the other hand, if we use the modified definition of an SM-allocation 3:

$${}^{\bullet}t_1 \cap {}^{\bullet}t_2 \neq \emptyset \Rightarrow B(t_1) = B(t_2) \in {}^{\bullet}t_1 \cap {}^{\bullet}t_2$$

we would have to call the following net "SMA":



"SMA"-net, yet unbounded



Only possible "SM-reduction"; it is a SCSM.

Figure 10

There are 2 SM-allocations, according to the unmodified definition: $B_1(t_2) = p_1$

$B_2(t_2) = p_2$

(B_1 and B_2 are equal at the other transitions.) But $^{\cdot}t_1 \cap {}^{\cdot}t_2 = \{p_1\}$, and thus B_2 is not compatible. Yet it is B_2 which reveals the bad SM-reduction. Thus, the modified SM-allocation and the resulting definition of "SMA" are not satisfactory.

On page 2 we suggested a "simple equivalence" which would transform the EFC net of Fig. 9 into the following FC net, which is SMA:

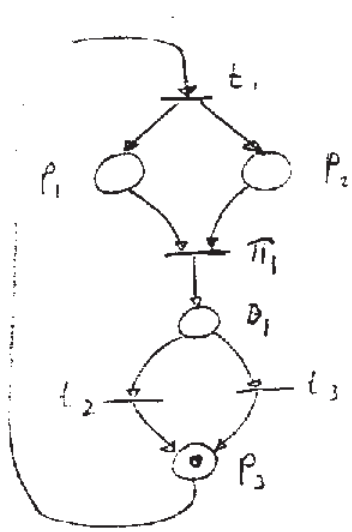


Figure 11

We have replaced the arcs of the complete subgraph from $P = \{p_1, p_2\}$ to $T = \{t_2, t_3\}$ by an additional λ-transition θ_1 and an additional blank place π_1 and the arcs $p \cdot \theta_1$ for each $p \in P$, the arcs $\pi_1 \cdot t$ for each $t \in T$, and the arc $\theta_1 \cdot \pi_1$. It is easy to see that, at least in the case of EFC nets, this transformation preserves liveness in addition to the equivalence properties of a transformation such as going to the released form. We shall see that this is also true if the result is SMA, but this in general it is not true (see Fig. 16). We therefore introduce the concept of <u>Extended</u> <u>SMA</u> to cover cases like Fig. 9 and still exclude cases like Fig. 10.

The net situation that leads to trouble is the existence of a complete bipartite subgraph from a subset of places, such as P, to a subset of transitions, such as T. A complete subgraph is also called a <u>clique</u>; therefore we shall call the transformation that yields the net in Fig. 12 a <u>clique-reduction</u>, and the net in Fig. 12 is the <u>clique-reduced form</u> of the net in Fig. 11.

<u>Def</u>: A <u>clique</u> in a Petri Net $\langle \Pi, \Sigma \rangle$ is a subnet $\langle P, T \rangle$ which is complete from P to T. This can be expressed as follows:

$$P \subseteq \Pi, \quad T \subseteq \Sigma : \quad (\forall p \in P)(\forall t \in T)\ p \cdot t$$

<u>Def</u>: A clique $\langle P, T \rangle$ is said to be maximal if it is not properly contained in any other clique.

<u>Def</u>: A clique $\langle P, T \rangle$ is trivial if $|P| \leq 1$ or $|T| \leq 1$.

A clique can be <u>reduced</u> in a way suggested by the example on page 2, Figure 3.

<u>Def</u>: Given a Petri Net $Q = \langle \Pi, \Sigma \rangle$ and a clique $\langle P, T \rangle$, we obtain a <u>clique-reduced form</u> Q' (with respect to $\langle P, T \rangle$) of Q in the following way: $(Q' = \langle \Pi', \Sigma' \rangle)$

- add a λ-transition θ : $\Sigma' = \Sigma \cup \{\theta\}$
- add a blank place π : $\Pi' = \Pi \cup \{\pi\}$
- add the arc $\theta \cdot \pi$
- bundle all arcs from P to T into θ and out of π, i.e:

There are two bijections $f: \Pi \to \Pi' - \{\pi\}$

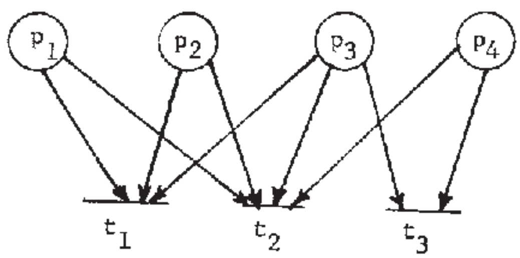
$g: \Sigma \to \Sigma' - \{\theta\}$ such that:

$$\left.\begin{array}{l}
\forall t \in \Sigma,\ \forall p \in \Pi:\ \ t \cdot p \ \Rightarrow\ g(t) \cdot f(p) \\
\forall p \in \Pi,\ \forall t \in \Sigma - T:\ \ p \cdot t \ \Rightarrow\ f(p) \cdot g(t) \\
\forall p \in \Pi - P,\ \forall t \in \Sigma:\ \ p \cdot t \ \Rightarrow\ f(p) \cdot g(t) \\
\forall p \in P,\ \forall t \in T:\ \ f(p) \cdot \theta \ \ \&\ \ \pi \cdot g(t) \\
\qquad\qquad\qquad\qquad\qquad \theta \cdot \pi
\end{array}\right\}\ \text{arcs in the reduced Net}$$

Thus, after a clique $\langle P, T\rangle$ has been reduced, no arc goes directly from any place in P to any transition in T:

$$\text{before:}\quad P \subseteq {}^{\cdot}T \quad \& \quad T \subseteq P^{\cdot}$$
$$\text{after:}\quad P \cap {}^{\cdot}T = \emptyset \quad \& \quad T \cap P^{\cdot} = \emptyset$$

Example: The following Petri Net contains several non-trivial cliques, some of which contain each other (i.e. not maximal), and some of which overlap (i.e. have at least one arc in common, and therefore both a place and a transition in common.) It is helpful to list all non-trivial cliques:



original net, Q_0

$C_1 = \langle\{p_1, p_2\},\ \{t_1, t_2\}\rangle$
$C_2 = \langle\{p_2, p_3\},\ \{t_1, t_2\}\rangle$
$C_3 = \langle\{p_1, p_2, p_3\},\ \{t_1, t_2\}\rangle$ (maximal clique)
$C_4 = \langle\{p_3, p_4\},\ \{t_2, t_3\}\rangle$ (maximal clique)

Figure 12

Now, there are four possible clique-reductions of Q_0, where reduction with respect to C_i yields Q_i:

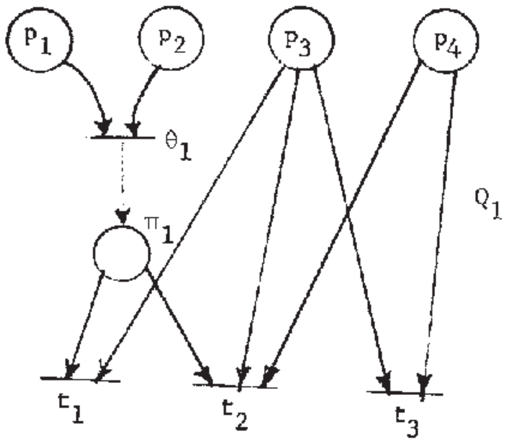


Figure 13a

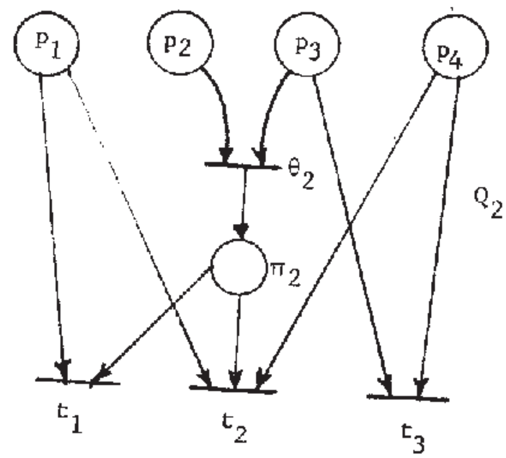


Figure 13b

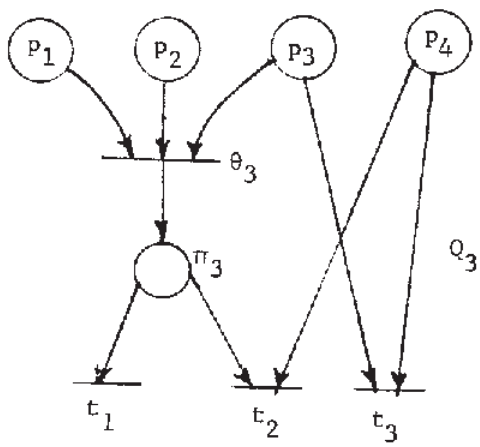


Figure 13c

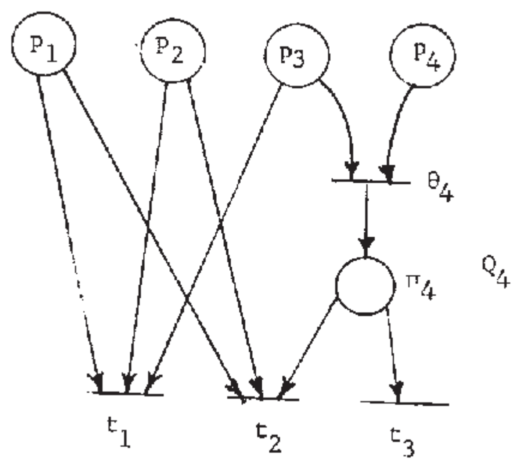


Figure 13d

Notice that Q_3 is clique-free: It may happen that the reduction of one clique destroys other, previously overlapping, cliques. Q_3, which is not further reducible, is called a completely clique-reduced form of Q_0. Q_2 and Q_4 each still contain one non-trivial clique. Further reduction of Q_2 and Q_4 yields $Q_{2.1}$ and $Q_{4.1}$, respectively (Figures 14a and 14b).

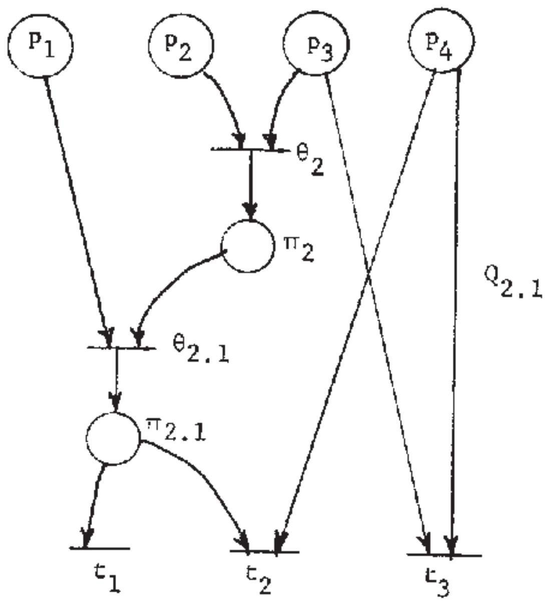


Figure 14a

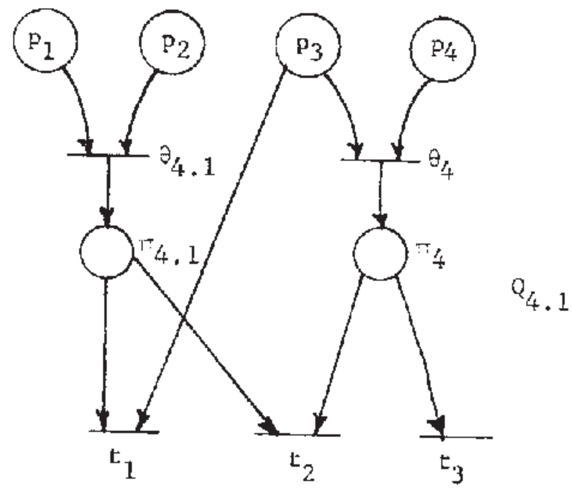


Figure 14b

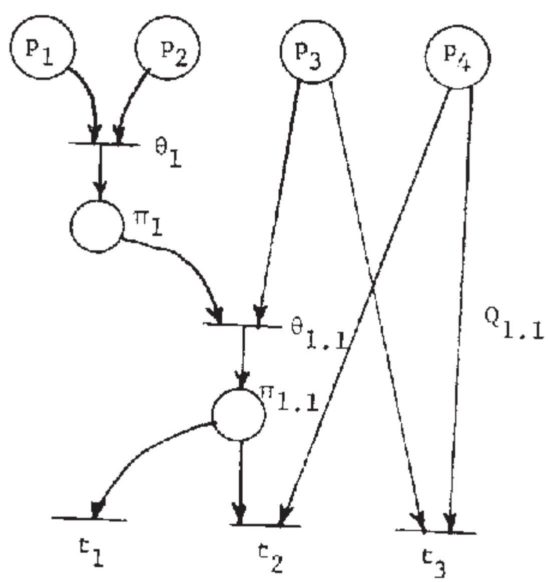


Figure 14c

Q_1 still contains two cliques, but reduction of either one will eliminate the other as well. Notice that one of them is actually C_4, which has not been affected by the first reduction. If we reduce C_4 now, we notice that we get the same net as $Q_{4.1}$ (up to the labeling of the additional vertices). The other clique in Q_1, viz. $\langle\{\pi_1,p_3\}, \{t_1,t_2\}\rangle$, has been introduced by the reduction of C_1, because C_1 was not maximal. Its reduction yields $Q_{1.1}$.

We conclude that a given Petri Net usually has many clique-reduced forms, and often also has distinct completely clique-reduced forms. Even if there are no overlapping maximal cliques, the reduction of non-maximal cliques often results in distinct completely clique-reduced forms.

If we only reduce maximal cliques, any completely clique-reduced form so obtained will be called a <u>maximally clique-reduced form</u> (<u>MCRF</u>). It is then clear that if there are no overlapping maximal cliques in the original net, then there will be a unique maximally clique-reduced form.

Now, we shall show that clique reduction has many properties in common with releasing a Petri Net. Indeed, it is easy to see that releasing an arc is the same as reducing a non-maximal trivial clique, namely the arc in question:

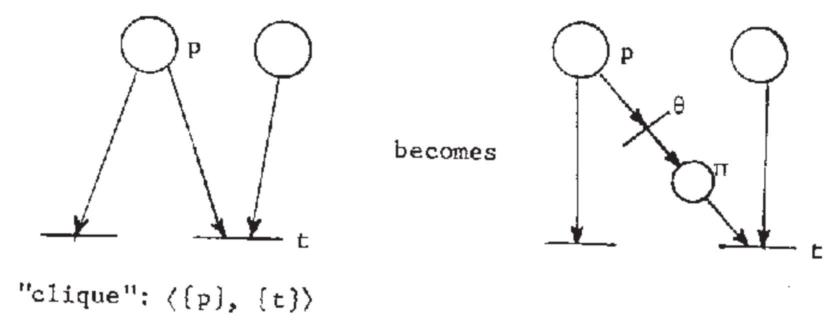


Figure 15

Thus, if we permit "reduction" of trivial cliques, among the infinity of clique-reduced forms (complete clique-reduction does not make sense anymore, since there will always be trivial cliques to reduce) we will find the released form of the net and of its clique-reduced forms. We will, therefore, restrict ourselves to the reduction of non-trivial cliques.

We can take advantage of this similarity: When proving facts about clique-reduced nets, we just mention the corresponding proofs about released forms. These proofs apply with very little modification to the reduction of a single clique. But then, a transitivity argument is used to extend the proof to all clique-reduced forms of a given net. That is because firing agreement is an equivalence relation over a given set of transitions, in this case the transitions of the original net, which are repeated in every reduced form.

Lemma 2: Let Q be a Petri Net, and Q_1 the result of reducing the clique $\langle P, T\rangle$, with additional vertices θ and π, as mentioned in the definition on p. 18.

(a) Let σ be a firing sequence in Q and σ' a firing sequence in Q_1 such that σ and σ' agree on the transitions of Q. Then the following property holds for the markings M and M' reached by σ and σ' in Q and Q_1, respectively:

$$\forall p \in P: \; M(p) = M'(p) + M'(\pi)$$
$$\forall p \notin P: \; M(p) = M'(p)$$

(b) The firing sequences of Q and Q_1 agree over the transitions of Q.

(c) If Q_1 is live then Q is live

(d) If Q_1 has a live marking then Q has a live marking.

(e) Q_1 is safe (bounded) iff Q is safe (bounded).

Proof: (a) The only difference between σ and σ' can be a firing of θ. Then notice, as in the proof of Theorem 2, that a firing of θ does not change $M'(p) + M'(\pi)$, where $p \in P$, that firings into P affect M(p) and M'(p) identically, and that firings out of P (in Q) affect M(p) exactly as the corresponding firing in Q' affects $M'(\pi)$, without altering M'(p). All other places are clearly affected the same way by σ and σ'.

(b,c,e): The proofs of Theorem 3 (page 12), Lemma 1 (page 14), and Theorem 5 (page 15) apply, where instances of $\vec{e}_j$ are replaced by a firing of θ, and where part (a) of this Lemma is used in lieu of Theorem 2.

(d) If Q_1 has a live marking, all we must prove that it has a live marking which could correspond to a marking in the original net, i.e. a marking which leaves the additional place π blank. Then (c) applies. But that is clearly the case, since π cannot be in a selfloop, by construction. If a given live marking class never empties place π, it means π contains a trapped token, and if we remove it, we get maybe a different marking class, but still a live one. (In a sense, if a token not in a self-loop is never removed by the firing sequences of a net, its removal by an outsider will not be "noticed" by the net. Only tokens in a self-loop can be "used" for firing without being "removed".)

<u>Theorem 8</u>: (a) The firing sequences of a Petri Net Q and any of its clique-reduced forms Q' agree over the transitions of Q.

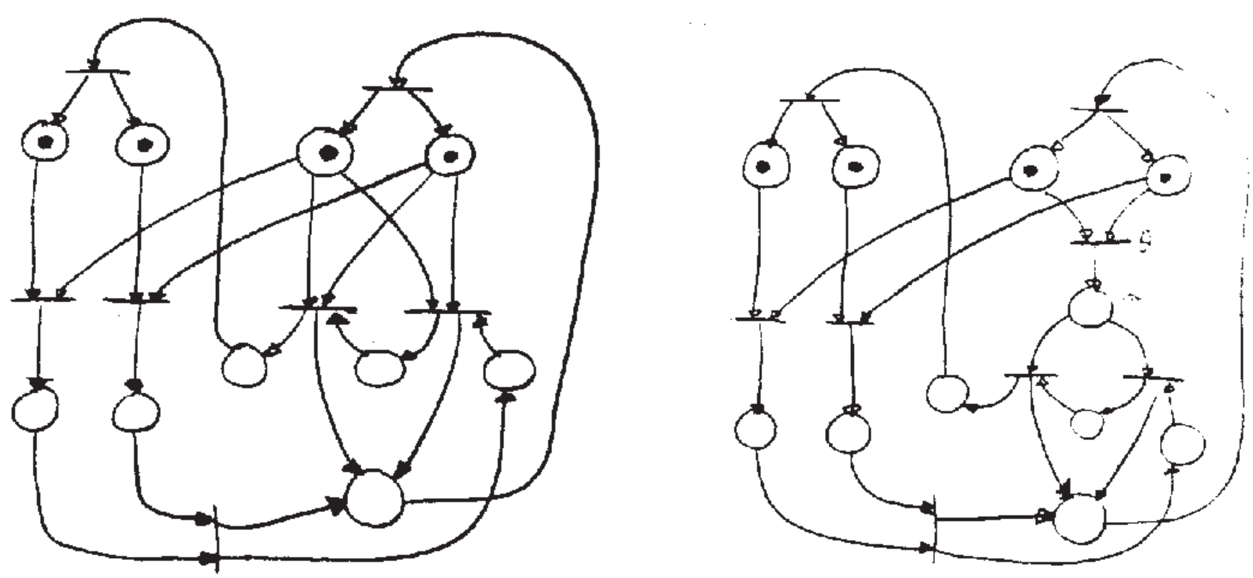
(b) If any clique-reduced form is live then the original net is live.

(c) If any clique-reduced form has a live marking, then the original net has a live marking.

(d) The safeness (boundedness) of any clique-reduced form implies the safeness (boundedness) of the original net, and conversely.

<u>Proof</u>: It follows from Lemma 2 by the transitivity argument. In particular, though some clique-reduced forms may have firing sequences that agree over a larger set of transitions, all agree over the original set of transitions.

The converse of Theorem 8b and c is not true in general. A counterexample is shown in Figure 16.



<u>original</u>
LS, not ESMA

<u>clique-reduced</u>
not live, not SMA

Figure 16

Before we introduce ESMA nets, we should familiarize us a little more with the various features the completely clique-reduced forms of a Petri Net can have.

The example in Figure 17 is EFC. We see that some complete reductions are FC and others are not. Only the maximally clique-reduced form of an EFC net is guaranteed to be FC. Actually, close inspection of Fig. 17(f) shows that it could not even be SMA.

Now, we may define Extended State-Machine Allocatable nets (ESMA).

Def: A Petri Net is said to be ESMA iff at least one of its clique-reduced forms is SMA.

(We could have concentrated on completely clique-reduced forms, since if a net contains a clique, it cannot be SMA, which should be clear from Fig. 9, page 16.)

We shall now show that, in some sense, clique-reduction does not affect the decomposition of a Petri Net into its SM-components or its MG-components.

We only have to consider one clique-reduction, of say the clique $\langle P, T\rangle$, adding a transition θ and a place π. Transitivity will prove the rest. Now, any SM-component in the original net which intersects the clique will contain exactly one place in P and every transition in T. In the reduced net, there is a corresponding SM-component which differs from the previous one only by the additional transition-place pair θ, π. The appearance of the difference is shown below, for the appropriate portion of an SM-component.

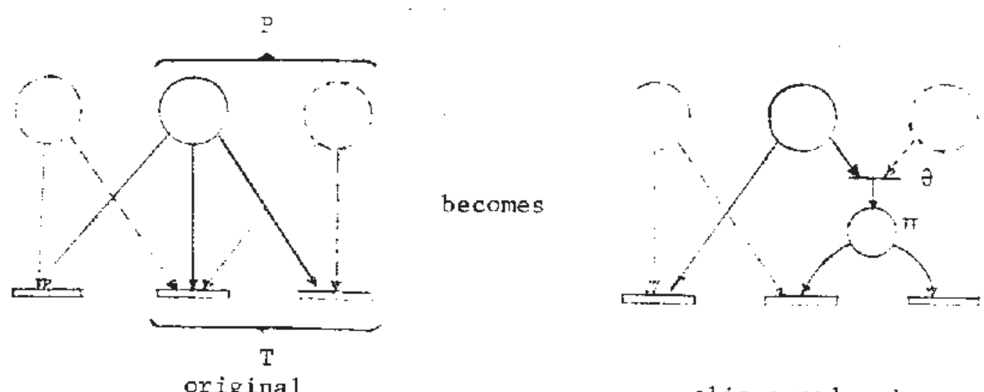


Figure 18

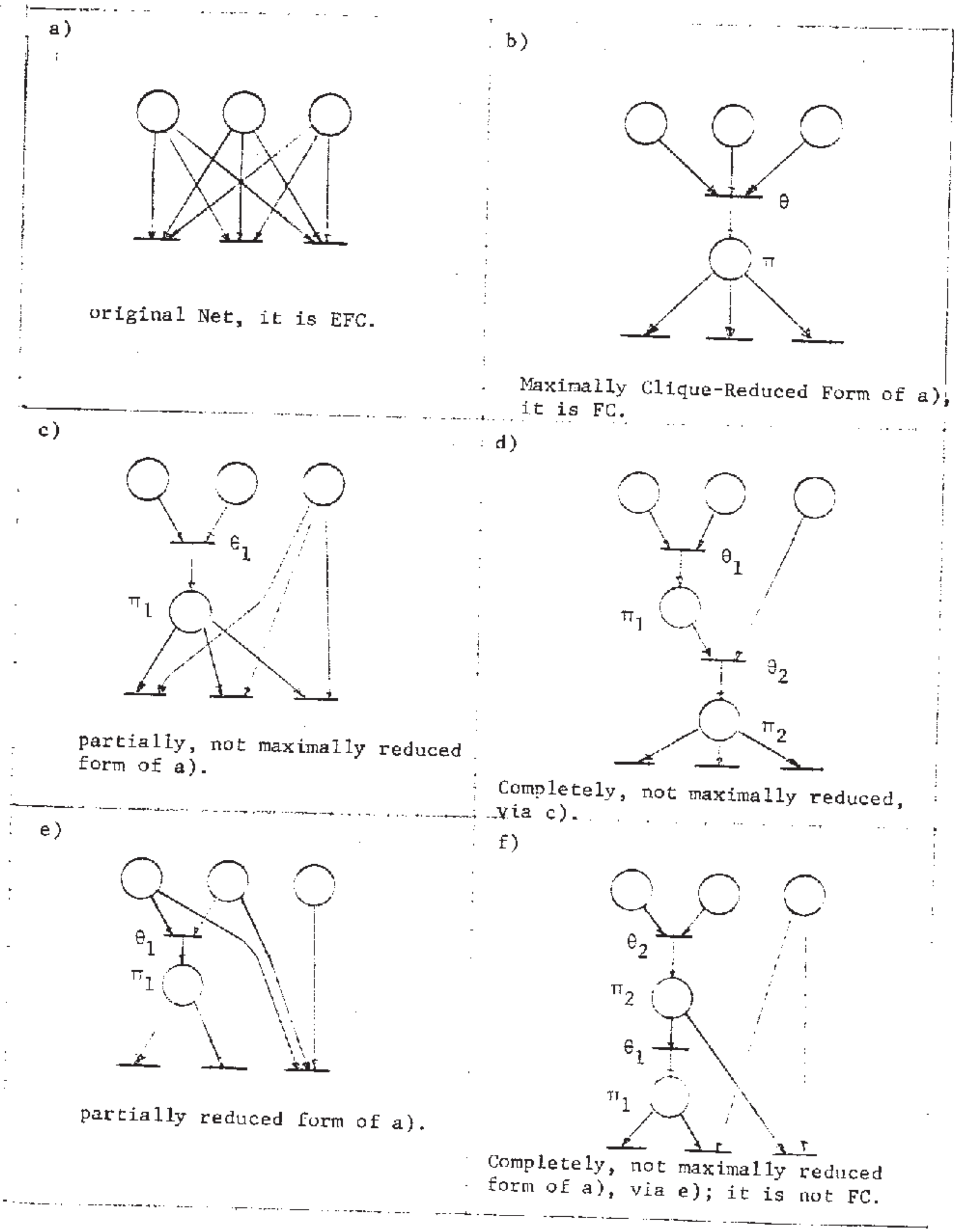

a)
original Net, it is EFC.
b)
θ
π
Maximally Clique-Reduced Form of a), it is FC.
c)
θ_1
π_1
partially, not maximally reduced form of a).
d)
θ_1
π_1
θ_2
π_2
Completely, not maximally reduced, via c).
e)
θ_1
π_1
partially reduced form of a).
f)
θ_2
π_2
θ_1
π_1
Completely, not maximally reduced form of a), via e); it is not FC.

Figure 17

In the case of MG-components, the correspondence is as shown below:

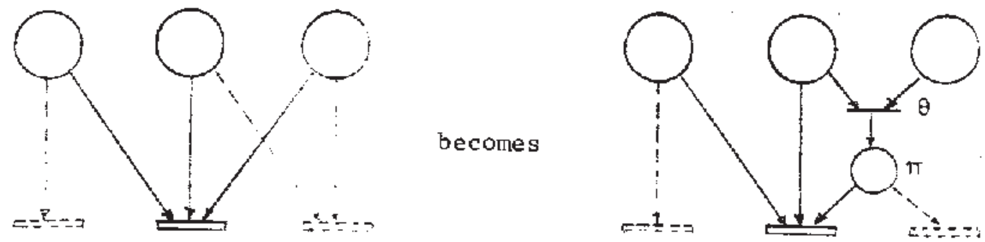


Figure 19

In both cases, it is clear that the original component is regenerated if we contract the arc containing the extra transition (in the case of SM-components) or place (in the case of MG-components). It is important to note that, whether we merge two transitions (in the MG) or two places (in the SM), the eliminated place π had a single input (from θ) and the eliminated transition θ had a single output (to π). This guarantees that the contraction does not change the connectivity of the component, and that it does not alter its possible firing sequences (up to transition θ, of course). It is in this sense that clique reduction does not affect the decomposition into SCSM's or SCMG's. We may also observe that, for both MG-components and SM-components, the firing sequences of the original component agree over the original transitions with the firing sequences of the corresponding component of the reduced net. Also note that the same is true in the case of components of the released form of a net; refer to the remark on page 22.

The preceding argument can be expressed more clearly by defining the operation of arc <u>contraction</u>, which is the inverse of clique-reduction and releasing.

<u>Def</u>: (a) An arc $t_o \cdot p_o$ is said to be <u>contractable</u> iff

$$t_o^{\cdot} = \{p_o\} \text{ \underline{and} } {}^{\cdot}p_o = \{t_o\} \text{ \underline{and} } {}^{\cdot}(p_o^{\cdot}) \cap {}^{\cdot}t_o = \emptyset$$

(this also implies $p_o^{\cdot} \cap ({}^{\cdot}t_o)^{\cdot} = \emptyset$)

(b) The <u>contraction</u> of a contractable arc $t_o \cdot p_o$ in a Petri Net $Q = \langle \Pi, \Sigma \rangle$ yields a Petri Net $Q' = \langle \Pi', \Sigma' \rangle$ which corresponds to Q in the following way: There are bijections $f: (\Pi - \{p_o\}) \to \Pi'$

$g: (\Sigma - \{t_o\}) \to \Sigma'$ such that

$$(p \in {}^{\cdot}t_o \ \& \ t \in p_o^{\cdot} \Rightarrow f(p) \cdot g(t)) \ \& \ p \cdot t \Rightarrow f(p) \cdot g(t)$$

$$t \cdot p \Rightarrow g(t) \cdot f(p)$$

In other words, the pair t_o, p_o is replaced by the clique $\langle {}^{\bullet}t_o, p_o^{\bullet} \rangle$.

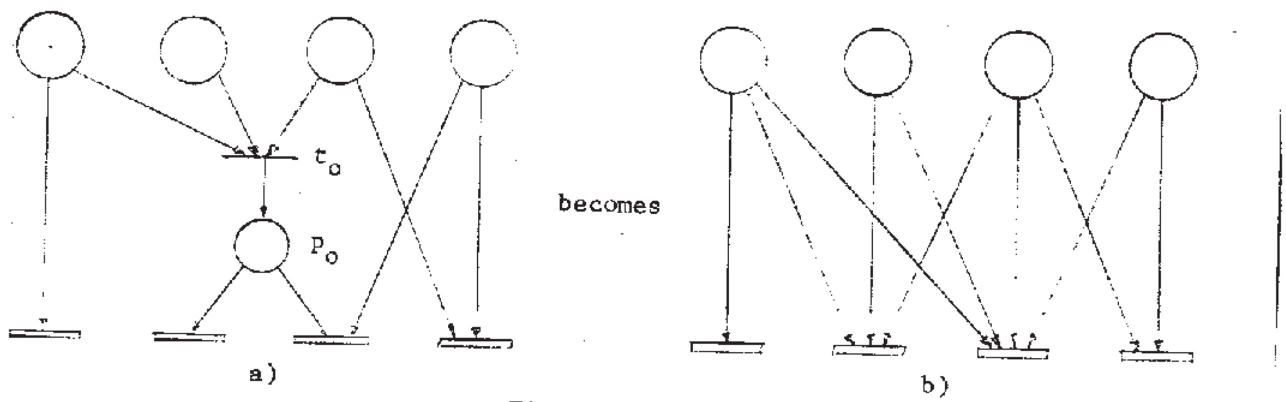


Figure 20

It follows directly from the definitions of clique reductions and contractions that:

Corollary: Q' is obtained from Q by reducing a clique ⟨P,T⟩ iff Q is obtained from Q' by contracting the arc $\theta \cdot \pi$ introduced while reducing the clique.

Now we can express the relationship between SM-components in the clique-reduced net and the corresponding components in the original net as follows:

Theorem 9: Let Q be a Petri Net, and let Q' be obtained from Q by clique-reduction and/or releasing.

(a) The reduction of any given clique or the releasing of any given arc introduces a contractable arc $\theta_i \cdot \pi_i$.

(b) Q can be obtained from Q' by contracting all arcs $\theta_i \cdot \pi_i$.

(c) The SCSM (MG)-components of Q are exactly obtained by contracting all arcs $\theta_i \cdot \pi_i$ in the corresponding SCSM (MG)-components of Q'.

Proof: (a) and (b) follow directly from the definitions.

(c) expresses what was discussed more informally on pages 26 and 27.

Since arc-contraction is the reverse of clique-reduction or releasing, we may also mention the following theorem as a consequence of Theorems 3, 4, 5 and 8:

Theorem 10: Arc contraction preserves liveness, safeness, boundedness, and firing sequences agreement over the remaining transitions.

This now permits us to state our main Theorem:

Theorem 11: (a) A Petri Net Q is ESMA iff it can be obtained from a WFFC Net Q' by repeated contractions.

(b) The firing sequences of Q and its SCSM(MG)-components agree (over Q) with those of Q' and its corresponding SCSM(MG)-components, respectively.

(c) Every ESMA Net has an LS marking, at which it is covered by one-token SCSM's.

Proof: It follows from Theorems 9 and 10 and the Live-and-Safeness Theorem for FC Nets.

Remarks about clique-reduction:

(a) Theorem 11a offers a very simple alternate definition for ESMA Nets: Those nets that can be obtained from a WFFC Net by arc contractions. But this does not offer a test to decide whether a given Net is ESMA. On the other hand, the definition given on page 25 carries with it such a test: Find all clique-reduced forms. But a much more convenient definition would be one which indicates which clique-reductions should be performed to get an SMA Net. A natural possibility would be to consider only Maximally Clique-Reduced Forms. Unfortunately, a given Petri Net may have several distinct Maximally Clique-Reduced Forms (MCRF) if there are overlapping maximal cliques.* However, we conjecture the following:

Conjecture: (a) Every MCRF of an ESMA Net is SMA.

(b) Every ESMA Net has a unique MCRF.

If this conjecture is true, it is sufficient to generate any MCRF of a Net and check whether it is SMA to decide whether the Net is ESMA.

(b) Figure 16 on page 24 shows that the "simple equivalence" mentioned on page 2 can be quite misleading, since it may not preserve liveness. However, if, for every clique $\langle P,T \rangle$ we reduce, we introduce a "return" λ-transition $\bar{\theta}$ in addition to θ and π, such that $\bar{\theta}^{\cdot} = P$ and $\pi \cdot \bar{\theta}$, we

* The reader may verify that the Net in Figure 20b has two distinct MCRF's, one of which is the one in Figure 20a.

get what we call a λ-strongly connected MG-component which preserves liveness. This kind of reduction we call <u>strong clique-reduction</u>. Everything that holds for ordinary clique reduction also holds for strong clique-reduction, except for the following modifications:

- Strong Clique-Reduction preserves liveness, i.e. if Q' is a SCRF (Strongly-Clique-Reduced Form) of Q, then Q live $\Leftrightarrow$ Q' live.
- There are additional SCSM-components, namely the λ-strongly connected MG's of the form $\langle P, \{\theta, \bar{\theta}\}\rangle$.
- Arc contraction may introduce multiple self-loops of $\bar{\theta}$-transitions, of the form $\langle P, \bar{\theta}\rangle$ where $\bar{\theta}^{\cdot} = {}^{\cdot}\bar{\theta} = P$; these can, of course, be eliminated without any change in marking class or firing sequences (disregarding $\bar{\theta}$).
- Strong Clique-Reduction introduces <u>non-promptness</u>, i.e. unboundedly long λ-firing sequences may occur between firings of original transitions.

As an example, Figure 21 shows an ESMA net, and Figure 22 shows its Maximally Strongly Clique-Reduced Form.

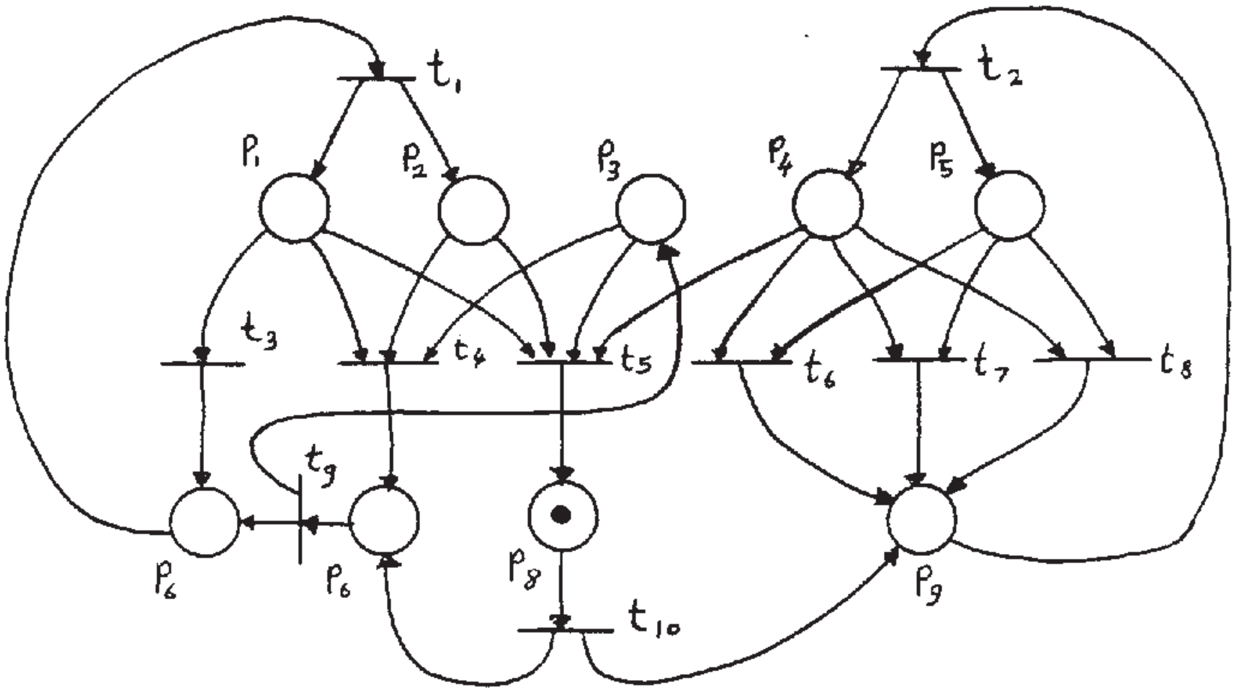
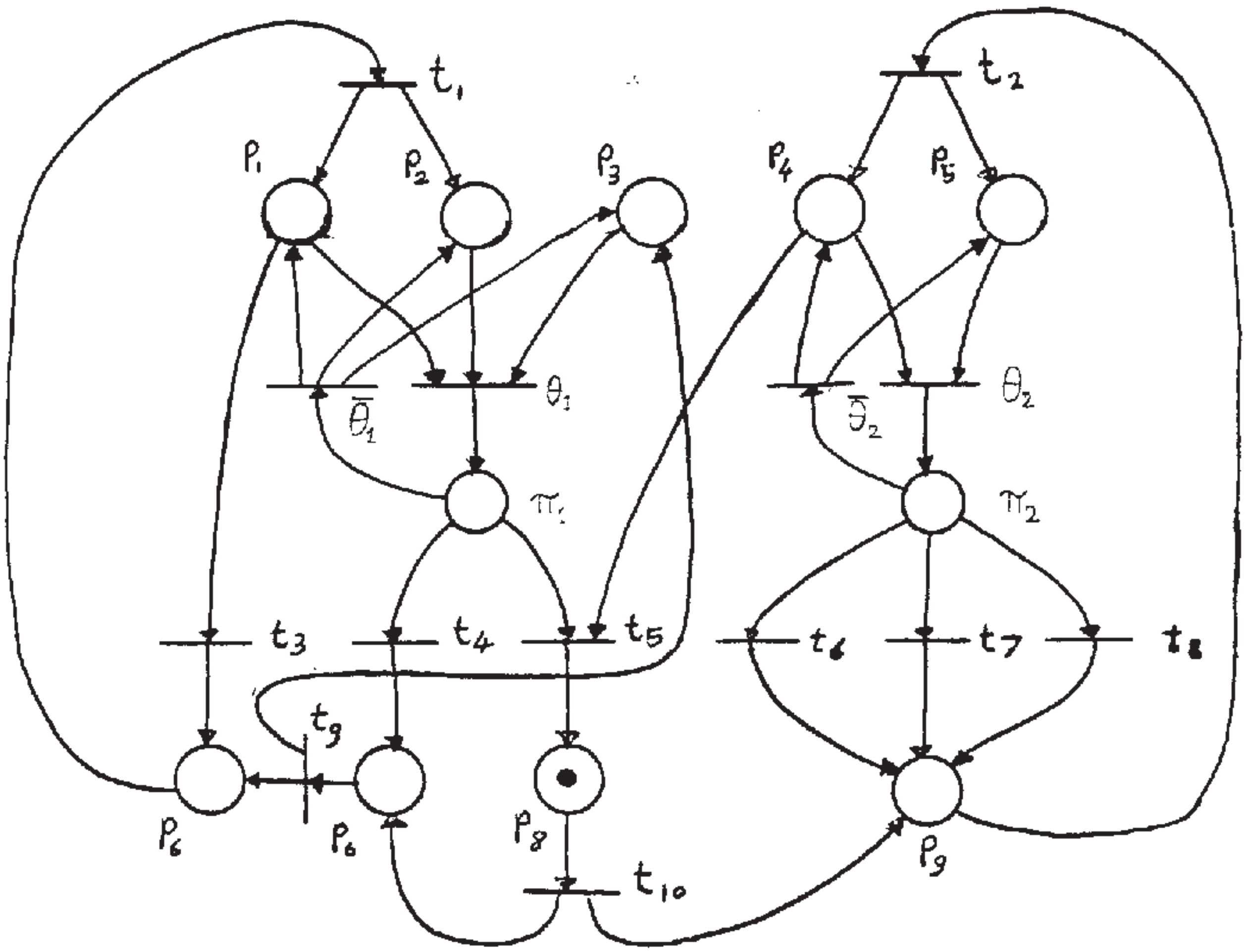


Figure 21



$$\begin{bmatrix} P_1 = \{p_1, p_2, p_3\} \\ T_1 = \{t_4, t_5\} \end{bmatrix} \qquad \begin{bmatrix} P_2 = \{p_4, p_5\} \\ T_2 = \{t_7, t_7, t_8\} \end{bmatrix}$$

(The two non-trivial maximal cliques)

Figure 22

E. Conclusion: Why ESMA Nets?

The prime motivation was to study Free-Choice like behavior in nets. If one interprets "Free Choice behavior" as meaning that at every place output, a token can choose which way to go without any consideration of the marking of other places, then non-FC SMA nets certainly qualify. The difference between an FC net and an SMA net is that, in the former, the token can fire a transition immediately after having made a choice, whereas in the latter, the token can fire such a transition eventually. In either case, every choice is always consistent with an LS marking. This is not so in non-SMA nets: it is possible to hang up if some token stubbornly insists on going left, though going right would be possible and indeed correspond to a firing sequence in an LS marking class.

In EFC nets, the tokens cannot always decide independently (EFC & $\neg$FC $\Rightarrow$ $\neg$SMA), but they "decide unanimously" in groups corresponding to maximal cliques. The group decisions are independent, however. Then, ESMA nets correspond to EFC nets as SMA nets correspond to FC nets. The clique reduction is just the Petri Net way to express the unanimous decision of a clique.

This close resemblance of ESMA nets to FC nets permits the extension of most of the results already established for FC nets.

As an example, we shall prove that the firing sequences of an LS ESMA net agree with those of every component one-token SCSM.

Lemma 4: In an LS FC net $Q = \langle \Pi, \Sigma \rangle$, let $Q' = \langle \Pi', \Sigma' \rangle$ be a component one-token SCSM. ($\Pi' \subsetneq \Pi$; $\Sigma' \subsetneq \Sigma$). Then the firing sequences of Q and Q' agree over Σ'.

*Live-and-Safeness Theorem of FC nets; see MAC TR-94, p. 62.

Proof: (a) Let σ be a firing sequence of Q and $\sigma' = \sigma \cap \Sigma'$ a matching sequence of Q'. Then the effects of σ and σ' on the marking of Q' are the same, i.e. $M_0[\sigma\rangle/\Pi' = M'_0[\sigma'\rangle$, where $M'_0 = M_0/\Pi'$. This is because Q' is a closed subnet of Q: If σ fires t and $t \notin \Sigma'$, then the firing of t does not change the marking of Q' (in Q) because $({}^{\cdot}t \cup t^{\cdot}) \cap \Pi' = \emptyset$. On the other hand, if $t \in \Sigma$, then it is firable in Q only if it is firable in Q', and the effect on Q' alone is the same as on Q' within Q.

(b) Suppose there exists a firing sequence of Q which does not agree with some firing sequence of Q'. Then there is a shortest such sequence $\sigma_1 t$, where: $\sigma_1 t$ firing sequence of Q

$t \in \Sigma'$

${\sigma'}_1 = \sigma_1 \cap \Sigma'$ is a firing sequence of Q'. Since t is firable at $M_0[\sigma_1\rangle$ in Q, we have: $(M_0[\sigma_1\rangle)({}^{\cdot}t) \geq 1$

But then, since $M_0[\sigma_1\rangle/\Pi' = M'_0[\sigma'_1\rangle$, we have:

$$M'_0[\sigma'_1\rangle({}^{\cdot}t \cap \Pi') \geq 1.$$

Thus, t would be firable in Q, and $\sigma'_1 t$ would agree with $\sigma_1 t$, contrary to our assumption.

(c) Suppose there exists a firing sequence of Q' which does not agree with any firing sequence of Q. There must be a shortest such sequence, $\sigma'_1 t$, where:

$\sigma'_1 t$ is a firing sequence of Q'

$\forall \sigma$ firing sequence of Q: $\sigma \equiv \sigma'_1 \bmod \Sigma' \Rightarrow$ t not firable at $M_0[\sigma\rangle$.

Let σ_1 be such a sequence: $\sigma'_1 = \sigma_1 \cap \Sigma'$ & t not firable at $M_0[\sigma_1\rangle$. But, by assumption, t is firable in Q' (removed from Q) at $M'_0[\sigma'_1\rangle$. Therefore, $M'_0[\sigma'_1\rangle\ ({}^{\cdot}t \cap \Pi') \geq 1$ (actually, it is equal to 1 since Q' is a one-token SCSM). Now, because of part (a) of this proof, we must also have:

$$M_0[\sigma_1\rangle({}^{\cdot}t \cap \Pi') = 1$$

Since t is not firable at that marking, there must be another, blank place in ${}^{\cdot}t$. But now, since Q is FC, the marked place ${}^{\cdot}t \cap \Pi'$ has no other output than t, and the unique token of Q' is trapped: no transition in Σ' can fire without t firing first. Thus every firing sequence of Q with initial marking $M_0[\sigma_1\rangle$ agrees with Q', but cannot fire t. This contradicts our assumption of liveness. See Figure 23.

QED

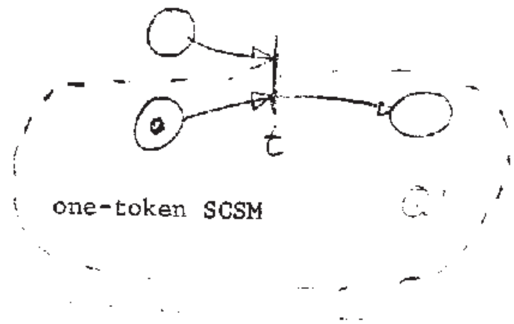


Figure 23

Note that the condition that Q' be a <u>one-token</u> SCSM is critical. The FC net (actually an MG) in Fig. 24 contains a 2-token SCSM (simple circuit) whose possible firing sequences include acbbaa... but the firing sequences of the full MG are strictly abcabc ...

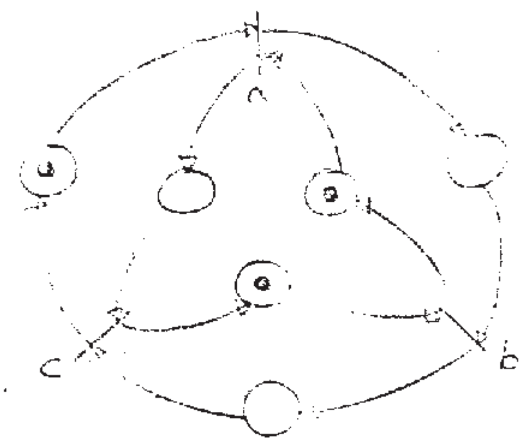


Figure 24

Also note that the situation is very different for SCMG-components. First, it is clear that, in an LSFC net, every firing sequence of a component SCMG <u>is</u> a firing sequence of the FC net. On the other hand, the following example (Figure 25) shows an LSFC net (actually an SCSM) and a firing sequence which agrees with neither component SCMG (simple circuit). But it is true that one can view a firing sequence of an LSFC net as containing successive cycles of different SCMG's, some of them concurrently ("shuffled" sequence), and possibly several cycles of one for one cycle of another. The example treated in detail in Chapter 6 of MAC TR-94 provides a good example; the reader may wish to analyze in these terms the firing sequence " a d j m i c k l m h i a c l b f e k f m g h " starting at a marking that puts one token on A and one token on H.*

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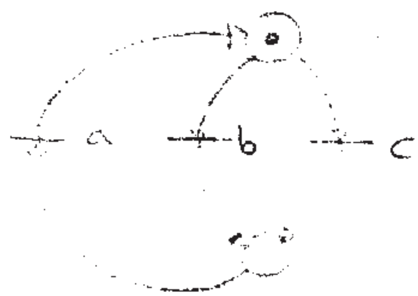
*Refer to the net on page 93 in MAC TR-94.



Component SCMG's: $\Sigma_1' = \{a,b\}$
$\Sigma_2' = \{a,c\}$

Firing sequence agreeing with neither:
σ = bacaba ... $\sigma \cap \Sigma_1'$ = baaba;
no good, for example

Figure 25

Theorem 12: In an LS ESMA net $Q = \langle \Pi, \Sigma \rangle$, let $Q' = \langle \Pi', \Sigma' \rangle$ be a component one-token SCSM. $(\Pi' \subseteq \Pi, \Sigma' \subseteq \Sigma)$. Then the firing sequences of Q and Q' agree over Σ'.

Proof: From Theorem 11 it follows that the WFFC Net Q_1 from which Q is obtained by contraction has the same firing sequences up to λ-transitions as Q, and has the same SCSM components, up to additional contractable places and λ-transitions. And since, in the component SCSM's, the λ-transitions always come with one of their associated original transitions (see Fig. 26), Lemma 4 applies and carries over to the original net Q.

QED

What has been said about SCMG components in LSFC nets carries over similarly.

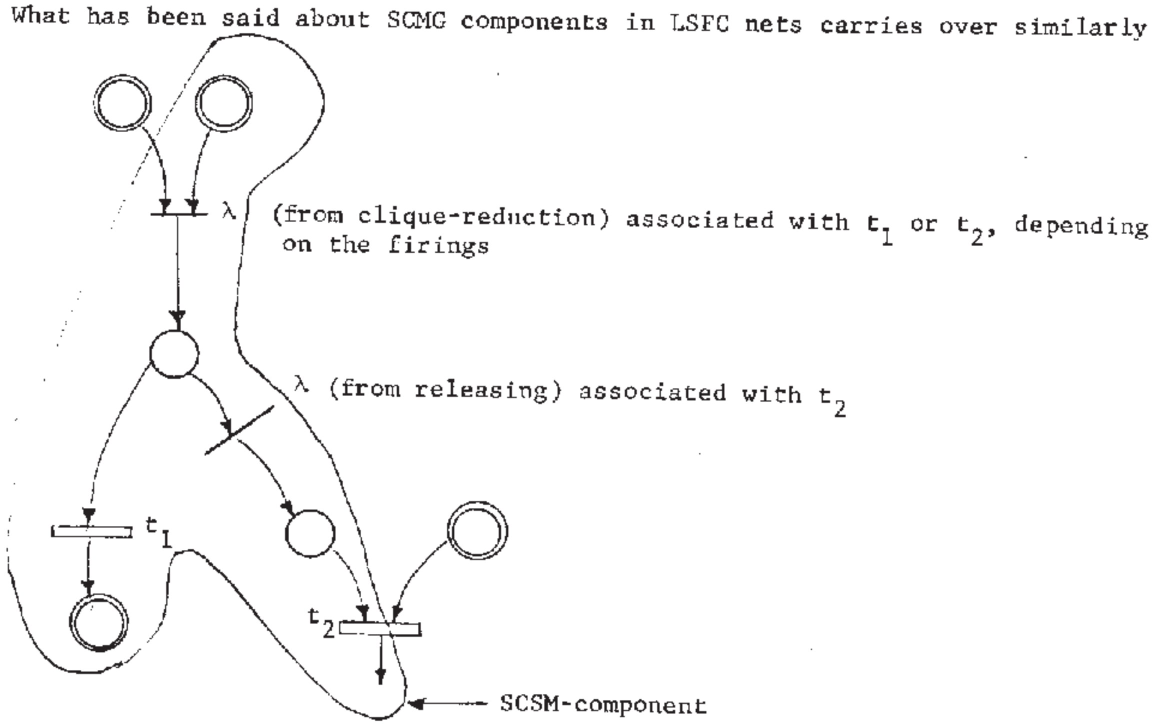


Figure 26

At one point, it was thought that ESMA nets would be precisely those nets for which the LS firing sequences agree with those of their component SCSM's. That this is not so is shown by the net in Fig. 27.

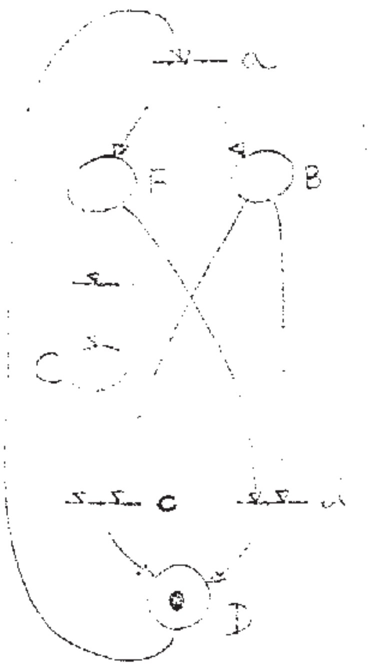


Figure 27

This net is not ESMA, not SMA, but it is LS and is the union of two 1-token SCSM's

$$Q_1' = \langle \{A, C, D\}, \{a, b, c, d\} \rangle$$

$$Q_2' = \langle \{B, D\}, \{a, c, d\} \rangle$$

Every firing sequence agrees with Q_1' and Q_2'. A similar situation arises in the non-maximally clique-reduced form of an EFC Net in Fig. 17f. This suggests that a larger class of Nets satisfying Theorem 12 is all live Petri Nets which can be contracted to ESMA Nets. But all symmetric Petri Nets also satisfy Theorem 12. (A Net is called symmetric if for every transition t there is a reverse transition $\bar{t}$ such that $\bar{t}^{\bullet} = {}^{\bullet}t$ and ${}^{\bullet}\bar{t} = t^{\bullet}$).

Another class of nets for which many of the WFFC results apply* are Simple Nets where each minimal deadlock is a SCSM. But an SMA net is not necessarily Simple, and Figure 28 shows a Simple Net covered by SCSM's (and each minimal deadlock is a SCSM) which is not SMA. It is LS, but the firing sequences of the SCSM component are constrained by the other component to a strict alternation.

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* F. G. Commoner, Deadlocks in Petri Nets. Report CA-7206-2311, Applied Data Research, Inc., Wakefield, Mass., June 1972.

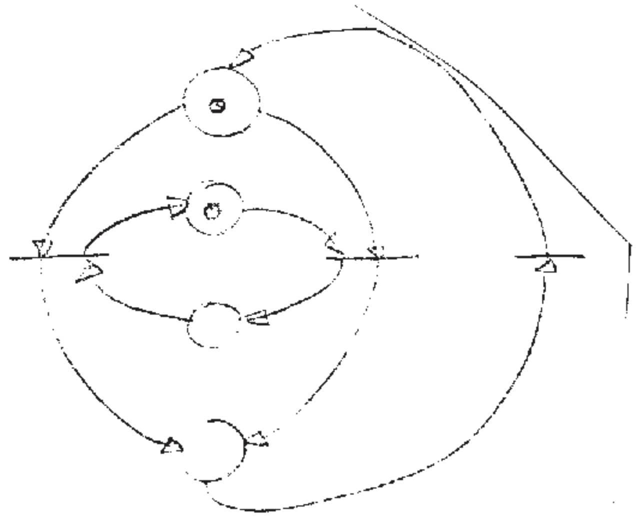
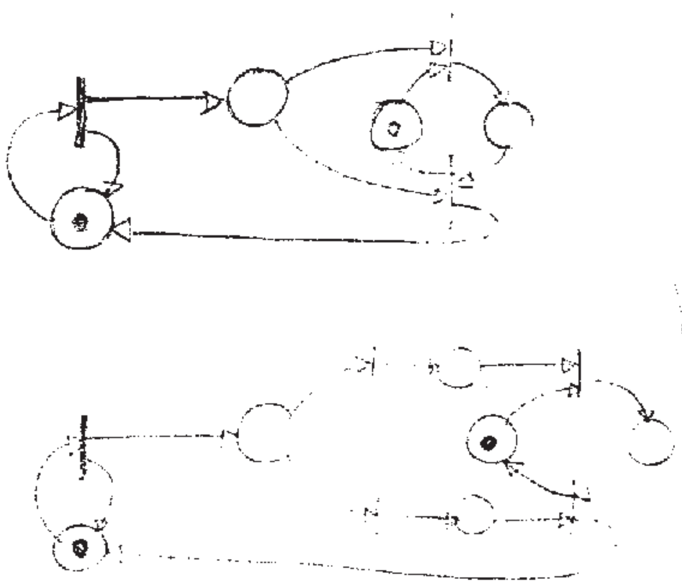


Figure 28

This concludes our study of State Machine Allocation and of the concepts of released form and clique reduction for bounded nets. Some questions:

- Are the results about agreeing firing sequences (Theorems 2, 3, 8) useful in general?
- Can anything of this be applied to unbounded nets? (See the limitation due to Theorem 7.)

Figure 29 shows a drastic example for Theorem 7, where the released form of a non-SMA net is live.



Both are live and unbounded, and their firing sequences agree.
Figure 29

References

F. Commoner, Deadlocks in Petri Nets. Report CA-7206-2311, Applied Data Research, Wakefield, Mass., June 1972.

F. C. Furtek, Modular Implementation of Petri Nets. S.M. Thesis, Department of Electrical Engineering, M.I.T., September 1971.

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