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Project MAC

Computation Structures Group Memo No. 46

Some Structural Properties of Demand Graphs

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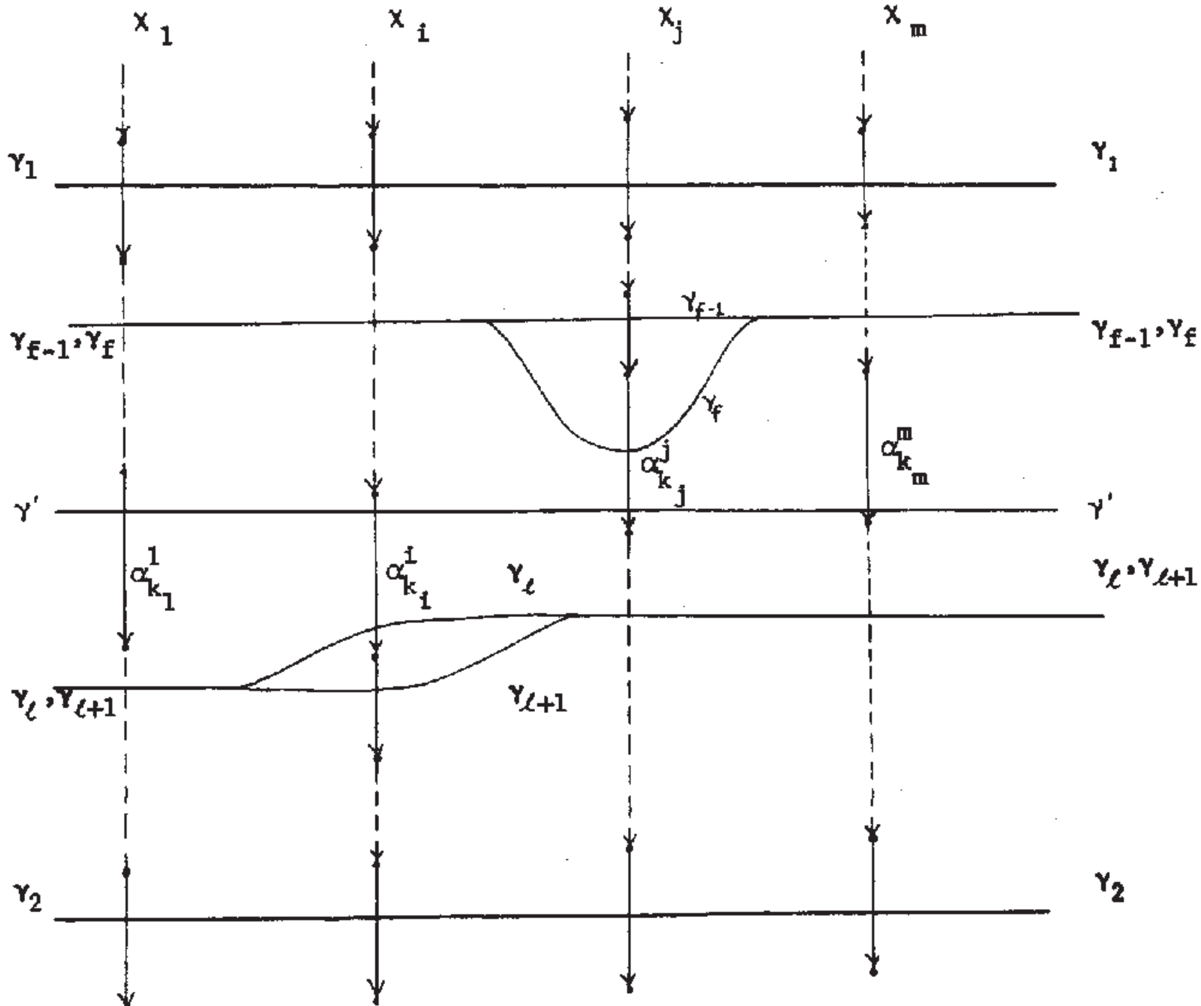

X_1
X_i
X_j
X_m
γ_1
γ_1
γ_{f-1}
γ_{f-1}, γ_f
γ_{f-1}, γ_f
γ_f
$\alpha^j_{k_j}$
$\alpha^m_{k_m}$
γ'
γ'
$\alpha^1_{k_1}$
$\alpha^i_{k_i}$
γ_ℓ
$\gamma_\ell, \gamma_{\ell+1}$
$\gamma_\ell, \gamma_{\ell+1}$
$\gamma_{\ell+1}$
γ_2
γ_2

Figure 1

This memo describes what may best be described as structural properties, as they relate to safeness, of the model introduced in [1]. The properties are expressed in the form of bounds on a general class of safeness tests. One conclusion is a feeling that the safeness algorithm of [1] is optimal.

Two lemmas and a theorem which will prove useful later will be proved first.

<u>Lemma 1.</u> Let γ_1 and γ_2 be two arbitrary feasible slices of a demand graph such that $\exists$ a path π from γ_1 to γ_2 consisting only of feasible slices (i.e. a <u>feasible sequence</u> exists from γ_1 to γ_2), and let $\alpha^1_{k_1}, \alpha^2_{k_2} \cdots \alpha^m_{k_m}$ be arcs such that $\forall_i \;\; \gamma_1 \cap X_i \trianglelefteq \alpha^i_{k_i} \trianglelefteq \gamma_2 \cap X_i$ and $\forall_i \not\exists$ an arc $\alpha^i_{s_i} \ni$

$$\alpha^i_{k_i} \trianglelefteq \alpha^i_{s_i} \trianglelefteq \gamma_2 \cap X_i \quad \wedge \quad d(\alpha^i_{s_i}) < d(\alpha^i_{k_i})$$

where "$\trianglelefteq$" means "occurs before or is the same as" and is defined on two arcs of a chain or two slices and $d(\alpha)$ is the demand associated with the arc α. Then there exists a feasible path π' from γ_1 to $\gamma' \equiv \alpha'_{k_1} \alpha'_{k_2} \cdots \alpha^m_{k_m}$.

<u>Proof</u>: Consider the path π. Let γ_f be the first slice in the sequence π that uses an arc lying between γ' and γ_2, and γ_ℓ be the last slice in π to use an arc belonging to γ'; that is# $\gamma_{f-1} < \gamma'$, $\gamma_f \not< \gamma'$ and $\gamma' \not< \gamma_\ell$, $\gamma' < \gamma_{\ell+1}$. Figure 1 shows these slices. Consider any slice γ on $\pi \ni \gamma_f \trianglelefteq \gamma \trianglelefteq \gamma_\ell$ and let γ^t be the transformed slice obtained from γ by replacing those components of γ that lie between γ' and γ_2 by the corresponding $\alpha^i_{k_i}$.

"$\gamma < \gamma^*$" means "$\gamma < \gamma^*$ and $\not\exists\, j \ni \gamma \cap X_j = \gamma^* \cap X_j$"

Then γ^t is feasible as the α's have the property that $d(\alpha^j_{k_j}) \leq d(\gamma \cap \chi_j)$ if $\alpha^j_{k_j} \leq \gamma \cap \chi_j \leq \gamma_2 \cap \chi_j$ and as γ is feasible, since it is an element of the feasible sequence π.

Thus by systematically transforming the slices in π from γ_{f+1} to γ_ℓ (inclusive) as above (and removing duplicate slices) one gets a sequence of feasible slices leading to γ' (since γ_ℓ when transformed becomes γ'). From another point of view the "moves" from γ_f to γ_ℓ are used to go from γ_f to γ' by ignoring all those moves which involve going across γ' on a chain. Q.E.D.

The remaining theorem and lemma have to do with a somewhat different concept, viz that of consistency. Can any pattern of feasibility and infeasibility over a lattice be actually obtained? That is, for a demand graph of fixed structure (number of chains and number of arcs per chain) can numbers be found for the arcs such that an arbitrary pre-specified pattern of feasibility and infeasibility of slices results? The tool for answering this question is the theory of linear inequalities. Each statement regarding the feasibility of a slice is a linear constraint of the form

$$\sum_{i=1}^{m} a^i_{r_i} \leq C$$

where $a^i_{r_i}$ is the concise notation for $d(\alpha^i_{r_i})$ and C is the capacity of the system.

Similarly infeasibility imposes a constraint of the form

$$\sum_{i=1}^{m} a_{r_i}^{i} > C \qquad \text{or} \qquad -\sum_{i=1}^{m} a_{r_i}^{i} < -C$$

As there are $\prod_{i=1}^{m}$ slices and only $\sum_{i=1}^{m} n_i$ arcs, where n_i is the number of arcs on the i^{th} chain there are likely to be more constraints than variables and this is why consistency becomes important. The following result from Cernikov [2] will be used in proving the two lemmas to follow:

"Theorem [3.4] Let $f_j(x) - a_j \le 0$ $j=1, 2, \cdots m$ be an arbitrary comparible system of linear inequalities over the linear space L(P), where P is an arbitrary ordered field, then the system

$$f_j(x) - a_j < 0 \qquad j = 1, 2, \cdots m'; \quad m' \le m$$

$$f_j(x) - a_j \le 0 \qquad j = m' + 1, \cdots m$$

is compatible iff the equation

$$\sum_{j=1}^{m} u_j \, f_j(x) = 0 \qquad \text{with the unknowns } u_1, \ldots, u_n$$

has no positive solutions satisfying the condition

$$a_1 u_1 + \cdots + a_m u_m = 0; \quad u_1 + \cdots u_{m'} > 0 \text{ "}$$

An intuitive understanding of the theorem can be obtained by rewriting the inequalities as

$$f_j(x) < a_j \qquad j = 1, 2, \cdots m' \quad m' \le m$$

$$f_j(x) \le a_j \qquad j = m' + 1, m' + 2, \cdots m$$

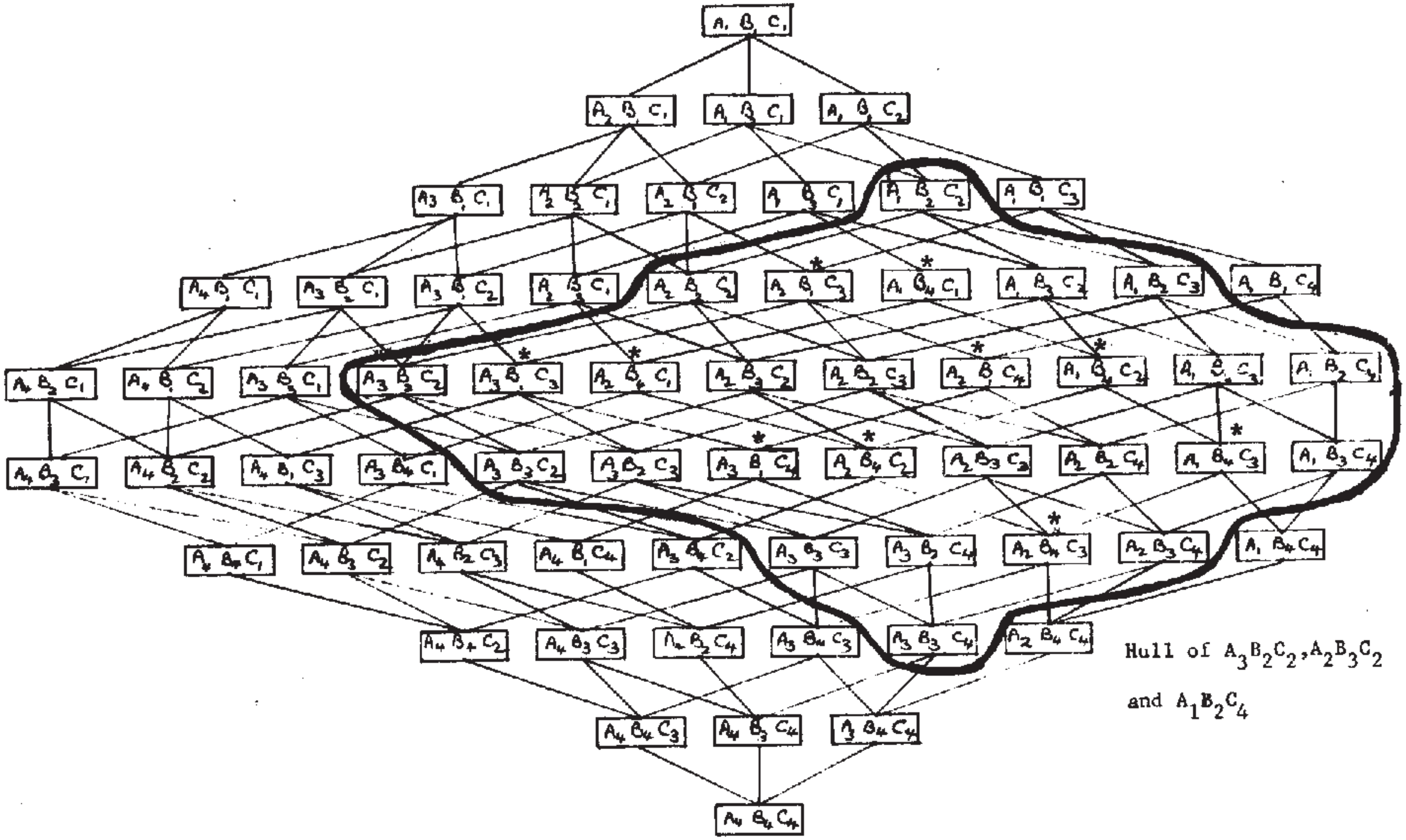
Here each $f_j(x)$ is of the form $\beta_1 x_1 + \beta_2 x_2 + \cdots \beta_n x_n$ where n is the dimension of the linear space L(P). Since multiplying an inequality by a positive constant does not alter it, it is clear that if positive multipliers u_j are found which (after multiplication) make the sum of the left hand sides identically zero, then in a compatible system the corresponding sum of the right hand sides must be greater than zero (unless no non-zero) multiplier multiplies the j^{th} inequality for $1 \le j \le m'$) or one gets the absurd conclusion $0 < 0$! What is less obvious, and therefore interesting, is that this condition is also sufficient for compatibility.

Since $\{\forall i, r_i \quad a^i_{r_i} = \frac{C}{m}\}$ is clearly a solution of the system of inequalities:

$$\left.\begin{array}{ll} & \sum_{i=1}^{m} a^i_{r_i} \le C \\ \text{and} & \\ & \sum_{i=1}^{m} a^i_{r'_i} \ge C \\ \text{or} & \\ & -\sum_{i=1}^{m} a^i_{r'_i} \le -C \end{array}\right\} \text{ in particular } \quad \begin{array}{l} \sum_{i=1}^{m} a^i_{r_i} = C \\ \\ \sum_{i=1}^{m} a^i_{r'_i} = C \end{array}$$

the system is compatible and the theorem above is applicable to the system resulting when the $\ge$ inequalities therein are changed to $>$.

<u>Definition</u>: The <u>hull</u> of a set A of slices is the set of all slices σ in the lattice which satisfy $g\ell b(A) \le \sigma \le \ell ub(A)$. Figure 2 shows the hull of a set of slices. By definition the hull of a set of slices in a lattice is a



Hull of $A_3B_2C_2$, $A_2B_3C_2$ and $A_1B_2C_4$

<u>Key</u>: Slices marked with an asterisk are not in the hull.

<u>Figure 2</u>

sub-lattice since the hull is closed with respect to the operations of taking the greatest lower bound and least upper bound of two slices. All slices in the hull of A thus lie on a path from ℓub(A) to gℓb(A).

<u>Theorem 1</u>. Consider a demand graph of m chains with n_i arcs on the i^{th} chain. Let $\gamma_1, \gamma_2, \cdots \gamma_p$ be slices required to be infeasible and $\gamma_{p+1}, \cdots \gamma_q$ be slices required to be feasible. Then a set of numbers (demands) for the arcs of the demand graph such that these conditions are met exist if none of the slices $\gamma_{p+1}, \cdots \gamma_q$ lies in the hull of $\gamma_1, \cdots \gamma_p$ (or symmetrically, none of the slices $\gamma_1, \cdots \gamma_p$ lies in the hull of $\gamma_{p+1}, \cdots \gamma_q$). In other words it is necessary that at least one of $\gamma_{p+1}, \cdots \gamma_q$ lie in the hull of $\gamma_1, \ldots \gamma_p$ for inconsistency.

<u>Proof</u>: The system of inequalities whose consistency is being examined is:

$$-\left(\sum_{i=1}^{m} a^i_{r_i}\right)_{\gamma_j} < -C \qquad 1 \le j \le p \text{ ---------------------------- } (1)$$

corresponding to the infeasible slices

$$\text{and } \left(\sum_{i=1}^{m} a^i_{r_i}\right)_{\gamma_k} \le C \qquad p+1 \le k \le q \text{ --------------------------- } (2)$$

corresponding to the feasible slices

where the $a^i_{r_i}$ are rational numbers.

<u>Step 1</u>: Let positive multipliers λ_j and μ_k for the two groups exist such that

$$-\lambda_1 \left(\sum_{i=1}^{m} a^i_{r_i}\right)_{\gamma_1} -\lambda_2 \left(\quad\right)_{\gamma_2} - \cdots \lambda_p \left(\quad\right)_{\gamma_p}$$

$$+ \mu_{p+1} \left(\sum_{i=1}^{m} a^i_{r_i}\right)_{\gamma_{p+1}} + \cdots\cdots + \mu_q \left(\quad\right)_{\gamma_q} \equiv 0$$

i.e. $- \lambda_1(\ ) - \lambda_2(\ ) - \cdots \lambda_p(\ ) \equiv \mu_{p+1}(\ ) + \mu_{p+2}(\ ) + \cdots + \mu_q(\ )$ -----(3)

Then for consistency $- C \sum_{j=1}^{p} \lambda_j + C \sum_{k=p+1}^{q} \mu_k$ be > 0.

unless $\forall j\ (\lambda_j = 0)$. If $\sum_{1}^{p} \lambda_j \geq \sum_{p+1}^{q} \mu_k$ then clearly the system is inconsistent.

Consider the identity in (3). It will be noticed that each term in parentheses contains exactly m variables, each with coefficient 1. Since one can multiply (3) through by the LCM of the λ's and μ's to get integer multipliers, it may be assumed that the λ's and μ's are integers. The number of terms appearing on the left hand side when (3) is expanded out is $m \sum_{1}^{p} \lambda_j$ while that on the right hand side is $m \sum_{p+1}^{q} \mu_k$. Since (3) is an identity, these two numbers must be equal.

Thus $\not{m} \sum_{1}^{p} \lambda_j = \not{m} \sum_{p+1}^{q} \mu_k$

Therefore, $-C \sum \lambda_j + C \sum \mu_k = 0$

The system is therefore inconsistent if positive integer λ's and μ's exist which satisfy (3). This can be reworded as saying that for consistency no permutation of the labels of slices in a selection (with repetition) from $\{\gamma_1, \gamma_2, \cdots \gamma_p\}$ should produce a permutation of the labels of slices in a selection of the same order from $\{\gamma_{p+1}, \cdots \gamma_q\}$ (again with repetition allowed). This follows because the $a^i_{r_i}$ correspond one to one with the $\alpha^i_{r_i}$ (label components of slices) and because a multiplier λ, (or μ) > 1 corresponds to picking the slice more than once.

<u>Step 2</u>: Now consider a selection (in this proof, always with repetition allowed) from $\{\gamma_1, \cdots \gamma_p\}$. Then any permutation of the labels of these slices must yield slice-labels with the property that each component $a^i_{r_i}$ satisfies

gℓb of the α^i components of the slices in the selection	$\leq \alpha^i_{r_i} \leq$	ℓub of the α^i components of the slices in the selection		where "$\leq$ means" earlier than or the same as" and the arcs on each chain are numbered in sequence downwards

so that gℓb of the slices from the selection $\leq$ the slice in question (resulting from a permutation) $\leq$ ℓub of the slices from the selection

i.e. the slice lies in the hull of $\{\gamma_1, \cdots \gamma_p\}$

For example one slice-label resulting from the permutation of $A_3\ B_2\ C_2$ and $A_2\ B_3\ C_2$ is $A_3\ B_3\ C_2$ which clearly satisfies $A_1\ B_2\ C_2 \leq A_2\ B_3\ C_2 \leq A_3\ B_3\ C_4$ and $A_2\ B_3\ C_2$ lies in the hull of $A_3\ B_2\ C_2$, $A_2\ B_3\ C_2$ and $A_1\ B_2\ C_4$

Thus if none of $\gamma_{p+1}, \cdots \gamma_q$ lies in the hull of $\gamma_1, \cdots \gamma_p$ then the labels of any selection from $\{\gamma_{p+1}, \cdots \gamma_q\}$ cannot be a permutation of the labels of a sub-set of $\{\gamma_1, \cdots \gamma_p\}$. Thus no inconsistency can result and therefore numbers for the arcs exist which satisfy both the feasibility and infeasibility requirements. Q.E.D.

<u>Lemma 2</u>. Consider a demand graph with m chains, the i^{th} chain having n_i arcs. Let $\{\gamma_1, \cdots \gamma_p\}$ be a set of slices of the demand graph which lie in one rank, R, and let $\gamma_1, \cdots \gamma_p$ be required to be infeasible (feasible). Furthermore let $\gamma_1, \cdots \gamma_p$ completely partition their hull (i.e. $\nexists$ a slice at the same rank as $\{\gamma_1, \cdots \gamma_p\}$ in the hull of $\{\gamma_1, \gamma_2, \cdots \gamma_p\}$ but $\notin \{\gamma_1, \cdots \gamma_p\}$. Now if all the slices in the hull above (below) rank R are feasible (infeasible) then a necessary and sufficent condition for inconsistency is the requirement that a slice in the hull below (above) rank R also be required to be feasible (infeasible).

<u>Proof</u>: The necessity follows from Theorem 1 above which states that any feasible nodes outside the hull do not affect the consistency or inconsistency. It remains to be shown that

i The feasibility of all nodes in the hull above rank R cannot cause inconsistency

ii The feasibility of any additional node which is below rank R causes inconsistency

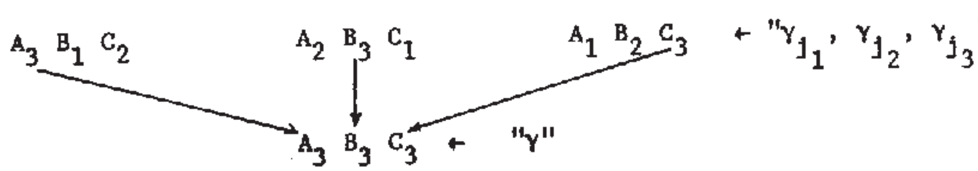
i The proof of this part consists merely in showing that a permutation of slice-labels of a subset of $\{\gamma_1, \ldots \gamma_p\}$ cannot result in labels of slices which lie entirely above or entirely below rank R. This follows from the fact that all slices in a rank have the same index-sum (when arcs on each chain in the demand graph are numbered sequentially from the top), representing the equal number of "moves" involved in reaching any of them from the top-most slice. Thus x nodes of rank R have a total index sum of $x \cdot R$ (if the arcs on a chain of the demand graph are numbered from 0 up). But $x \cdot R$ > the index sum of x nodes all at ranks less than R < the index sum of x nodes all at ranks greater than R, and it is known that any permutation of x slice-labels can yield only x slice-labels since each slice-label has to have a component corresponding to each chain of the demand graph.

ii Step 1. Let γ be a slice in the hull at a rank greater than R that is required to be feasible. Let $\{\gamma_{j_1}, \gamma_{j_2}, \ldots \gamma_{j_\ell}\}$ be a subset of $\{\gamma_1, \ldots \gamma_p\}$ such that γ is the gℓb of $\gamma_{j_1}, \ldots \gamma_{j_\ell}$

but not of $\{\gamma_{j_1}, \gamma_{j_2}, \dots \gamma_{j_\ell}\} - \gamma_j$ where γ_j is one of $\gamma_{j_1}, \dots \gamma_{j_\ell}$. That is $\{\gamma_{j_1}, \dots \gamma_{j_\ell}\}$ is the smallest set of slices at rank R whose gℓb γ is. Such a set has to exist since the set of all slices at rank R that lie on a path from ℓub $\{\gamma_1, \dots \gamma_p\}$ to γ certainly have γ as a gℓb and all such slices belong to the hull and hence to $\{\gamma_1, \gamma_2, \dots \gamma_p\}$ by assumption.

Now it will be shown that slices $\gamma_{s_1}, \dots \gamma_{s_{\ell-1}}$, all lying above rank R exist such that the labels of γ, γ_{s_1}, $\gamma_{s_2}, \dots \gamma_{s_{\ell-1}}$ are permutations of the labels of γ_{j_1}, $\gamma_{j_2}, \dots \gamma_{j_\ell}$.

In the discussion that follows the labels of slices γ, γ'... will be designated by γ, γ'... . This should cause no confusion as the context should resolve any ambiguity. $\gamma_{s_1}, \dots \gamma_{s_\ell}$ are obtained as follows: Take the elements that make up γ from γ_{j_1}, $\gamma_{j_2} \dots \gamma_{j_\ell}$. Take any of the ℓ "stripped" labels remaining and distribute its components among the other ℓ-1 stripped labels giving each a component from the same chain as the one it contributed to γ came from. The resulting labels are γ_{s_1}, $\gamma_{s_2}, \dots \gamma_{s_{\ell-1}}$. This is illustrated for $A_3\ B_1\ C_2$, $A_2\ B_3\ C_1$, $A_1\ B_2\ C_3$ below:



stripped labels: $B_1\ C_2$ $A_2\ C_1$ $A_1\ B_2$

result of distribution: $A_2\ B_1\ C_1$ $A_1\ B_2\ C_2$

clearly $(A_3 + B_1 + C_2) + (A_2 + B_3 + C_1) + (A_1 + B_2 + C_3)$

$$\equiv (A_3 + B_3 + C_3) + (A_2 + B_1 + C_1) + (A_1 + B_2 + C_2).$$

In general it is obvious that $\gamma, \gamma_{s_1}, \ldots \gamma_{s_{\ell-1}}$ is a permutation of $\gamma_{j_1}, \gamma_{j_2}, \ldots \gamma_{j_\ell}$.

It remains to be shown that $\gamma_{s_1}, \gamma_{s_2}, \ldots \gamma_{s_{\ell-1}}$ all lie above rank R, and are therefore feasible, which together with the feasibility of γ implies an incompatibility.

Step 2: It is obvious from the construction that each component of the cannibalized stripped-label is $\le$ the corresponding component of γ. In fact the above relation has to be strictly less than for at least one received component in each receiver γ'_{j_k} (i.e. stripped γ_{j_k}) viz the one component it alone can contribute to γ (if no such component exists γ_{j_k} is redundant in the set $\{\gamma_{j_1}, \ldots \gamma_{j_\ell}\}$). Thus each of the resulting slices $\gamma_{s_1}, \gamma_{s_2}, \ldots \gamma_{s_{\ell-1}}$ has an index sum $<$ that in rank R. (The reader can verify this in the example above.)

Q.E.D.

<u>Corollary 1</u>: If a slice γ of a demand graph is feasible but none of its immediate successors is, no slice other than γ in the hull of its successors can be feasible.

Corollary 2: If a feasible sequence of slices exists (in the lattice of slices of a demand graph) from a slice γ to rank L-m and the last slice in this partial sequence has m successors then the sequence can be extended to the bottom-most slice γ_T (at rank L) i.e. γ is safe.

This corollary follows from the fact that γ_T has always to be safe (by convention) and that the hull of m brothers extends to m-1 ranks below them.

The theorems and lemmas above and their corollaries represent tools which will be used in the rest of the discussion which relates to a class of safeness tests that are non-exhaustive but of a character different from the safeness algorithm of [1].

While the results in corollaries 1 and 2 of theorem 1 in [1] are valuable in simplifying the testing of safeness of slices, it is clear that the conditions of corollary 2 are infrequently met while those of corollary 1 may require completion of a sequence which is complete, in the sense that there is no chain but χ_1 remaining, so that no advantage is really gained. Consequently, the discussion that follows concerns itself with complete tests (i.e. ones that always yield an answer of "safe" or "unsafe") for safeness of a state σ of the kind that require the feasibility of p (from p=1 up to p="all possible") sequences of length k (some number). Can such tests be shortened (i.e. k reduced) when it is known that σ is an immediate successor of a safe allocation state? These tests will be called (k, p) feasibility tests. It is obvious that $\forall p(p \geq 1)$ an (L, p) feasibility test is always a safeness test, where L is the length of a path from γ to γ_T. Theorem 2 below concerns itself with (L, 1) feasibility tests and heorem 3 with (L, "all possible") feasibility tests; the bounds in these two theorems display the range for (L, p) feasibility tests.

<u>Theorem 2:</u> Consider an m-chain demand graph with a safe slice γ and a feasible immediate successor slice γ_i (corresponding to a move down chain χ_i). Let L be the relative rank of γ_T (the terminal slice) with respect to γ_i. Suppose there exists a feasible sequence from γ_i to a slice at relative rank K wrt γ_i.

Then (a) For $K < L-3$ safeness is not necessarily implied

(b) For $K \geq L-3$ safeness is necessarily implied

i.e. a (K, 1) feasibility test cannot be shortened beyond L-3.

<u>Proof</u>: It will be assumed in the proof that none of the following three special conditions in which γ_i is trivially known to be safe occur

i γ has no sons other than γ_i (A feasible path from γ to γ_T must pass through γ_i in this case.)

ii $d(\gamma_i \cap \chi_i) \leq d(\gamma \cap \chi_i)$ (In this case corollary 2 of Theorem 1 of [1] applies)

iii $\nexists$ j_1, j_2 integers $(1 \leq j_1, j_2 \leq m)$ such that $\gamma_i \cap \chi_{j_1}$ and $\gamma_i \cap \chi_{j_2}$ are not penultimate or terminal arcs of the corresponding chains.

For it is always possible to move from a penultimate arc to a terminal arc on a chain with a feasible slice resulting. When this is done at most one chain remains if the condition (iii) was not satisfied and it is known that it is possible to move down that chain by moves which result in feasible slices as d(arc) ≤ capacity for any arc. Consequently L ≥ 4.

Consider the slices at the rank L-n. Then n represents the total number of arcs remaining (i.e. below a slice γ_0 at this rank). The additive decomposition of n represents the manner in which these arcs are distributed over the m chains. In particular when n=3 there are only three distinct decompositions viz

$$3 = 3 + 0, \quad 3 = 2 + 1, \quad 3 = 1 + 1 + 1$$

Thus there are three forms for a slice label, viz

$$\gamma_1 \equiv \alpha^1_{n_1} \alpha^2_{n_2} \cdots \alpha^j_{n_j-3} \cdots \alpha^m_{n_m}$$ where n_i is the number of arcs on the i^{th} chain

$$\gamma_2 \equiv \alpha^1_{n_1} \alpha^2_{n_2} \cdots \alpha^{j_1}_{n_{j_1}-1} \cdots \alpha^{j_2}_{n_{j_2}-1} \cdots \alpha^{j_3}_{n_{j_3}-1} \cdots \alpha^m_{n_m}$$

$$\gamma_3 \equiv \alpha^1_{n_1} \alpha^2_{n_2} \cdots \alpha^{j_1}_{n_{j_1}-1} \cdots \alpha^{j_2}_{n_{j_2}-2} \cdots \alpha^m_{n_m}$$

Suppose a slice γ_0 at rank L-3 is feasible. If γ_0 is of the form γ_1 then clearly there is a possible path from γ_0 to γ_T as

d(arc) ≤ capacity for all arcs

= 0 for terminal arcs.

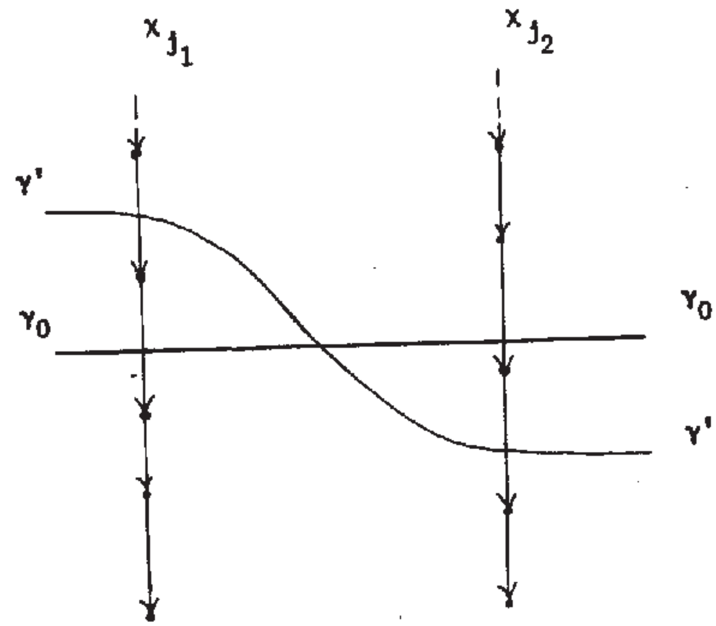


Figure 3

If it is of the form γ_2 again it is clear that there exists a feasible path from γ_0 to γ_T viz the one involving moves down the chains χ_{j_1}, χ_{j_2}, χ_{j_3} one at a time. Similarly, for form γ_3, moves down χ_{j_1} and χ_{j_2} provide a feasible sequence from γ_0 to γ_T. Consequently a feasible sequence from γ_i up to rank L-3 or greater is guaranteed to be extensible to γ_T. This proves part (b) of the statement of the theorem.

For n = 4, the distinct decompositions are n = 0+4, 4 = 1+1+1+1, 4 = 2+2, 4 = 3+1. On account of the third decomposition there are slices with labels of the form

$$\gamma_0 = \alpha^1_{n_1} \alpha^2_{n_2} \cdots \alpha^{j_1}_{n_{j_1}-2} \cdots \alpha^{j_2}_{n_{j_2}-2} \cdots \alpha^m_{n_m}$$

Suppose γ_0 is feasible. Then it does not necessarily follow that the two immediate successors of γ_0 are feasible. Consequently if there is a feasible sequence from γ of length L-4 which terminates on a slice of the form γ_0 then the sequence may or may not be extensible to γ_T so that the existence of the sequence of length L-4 cannot be used to draw conclusions regarding the safeness of γ_i. It remains to be shown that the infeasibility of the two immediate successors of γ_0, viz γ_1 and γ_2, does not conflict with the safeness of γ. Theorem 1 gave necessary conditions for incompatibility in terms of the hull of γ_1 and γ_2. Consider

$$\gamma' \equiv \alpha^1_{n_1} \alpha^2_{n_2} \cdots \alpha^{j_1}_{n_{j_1}-3} \cdots \alpha^{j_2}_{n_{j_2}-1} \cdots \alpha^m_{n_m} \quad (j_2 \neq i)$$

a slice distinct from γ_0 and yet at rank L-4. See figure 3 for the relationship of γ' to γ_0. Let γ' be accessible (by means of a feasible sequence) from γ.

Then the sequence $\gamma' \to \gamma'' \to \gamma''' \to \gamma'''' \to \gamma_T'$ is a feasible sequence from γ' to γ_T which does not intersect the hull of γ_1 and γ_2, where

$$\gamma'' \equiv \alpha^1_{n_1}\ \alpha^2_{n_2} \cdots \alpha^{j_1}_{n_{j_1}-3} \cdots \alpha^{j_2}_{n_{j_2}} \cdots \alpha^m_{n_m}$$

$$\gamma''' \equiv \cdots\cdots\cdots\ \alpha^{j_1}_{n_{j_1}-2}\ \cdots\cdots\cdots,$$

$$\gamma'''' \equiv \cdots\cdots\cdots\ \alpha^{j_1}_{n_{j_1}-1}$$

Consequently there exist paths accessible from γ going through rank L-4 to γ_T which avoid the hull of γ_1 and γ_2, i.e. γ_1 and γ_2 may be infeasible while γ is safe with no incompatibility.

It can be similarly shown that there are slices at ranks L-n ($n > 4$), viz.

$$\gamma_0 \equiv \cdots\cdots\ \alpha^{j_1}_{n_{j_1}-n'_{j_1}} \cdots\cdots\ \alpha^{j_2}_{n_{j_2}-n'_{j_2}} \cdots\cdots\ \alpha^{j_k}_{n_{j_k}-n'_{j_k}}$$

where $n = n'_j + n'_{j_2} + \cdots n'_{j_k}$ and each $n'_{j_i} \geq 2$, all of whose immediate successors may be infeasible while γ is safe (The slices corresponding to γ' and γ'' in the previous paragraph are

$$\gamma' \equiv \cdots\cdots\ \alpha^{j_1}_{n_{j_1}-(n'_{j_1}+1)} \cdots\cdots\ \alpha^{j_2}_{n_{j_2}-n'_{j_2}} \cdots\cdots\ \alpha^{j_k}_{n_{j_k}-(n'_{j_k}-1)} \cdots \quad \text{and}$$

$$\gamma'' \equiv \cdots\cdots\ \alpha^{j_1}_{n_{j_1}-(n'_{j_1}+1)} \cdots\cdots\ \alpha^{j_k}_{n-(n'_{j_k}-2)} \cdots\cdots$$

As it is not known at what form of slice a feasible sequence of length k from γ_i ends and it could be γ_0, no conclusions can be drawn about the safeness of γ_i when $k \leq L-4$. This is part (a) of the theorem. Q.E.D.

Theorem 3. Consider an m-chain demand graph with a safe slice γ and a feasible immediate successor slice γ_i (corresponding to a move down chain χ_i). Let λ be the second largest of the numbers $\{n_1-r_1-1, n_2-r_2-1, \cdots n_m-r_m-1\}$ where $\gamma_i \equiv \alpha^1_{r_1} \alpha^2_{r_2} \cdots \alpha^m_{r_m}$. Suppose all possible sequences of length k are feasible. Then

(a) For $k \geq \lambda$ determinacy of γ_i is a necessary consequence

(b) For $k \leq \lambda-1$ determinacy is not a necessary consequence

i.e. a (k, "all possible") feasibility test cannot be shortened beyond λ.

Proof: Once again the special cases singled out in Theorem 2 above will be ignored. Let $\lambda = n_\ell - r_\ell - 1$ then in addition if $n_i - r_i - 1 < \lambda$ then by the second corollary of Theorem 1 of [1] γ_i is known to be safe whenever $k \geq n_i - \gamma_i - 1$. Consequently it is assumed that $n_i - r_i - 1 \geq \lambda$.

If all sequences of length $k \geq \lambda$ are feasible then in particular the m-1 single chain sequences from γ_i to $[\gamma_i - \gamma_i \cap \chi_j] \cdot \alpha^j_{n_j-1}$ $(j \neq 2)$ are feasible. In fact as a consequence of the convention that $d(\alpha^j_{n_j}) = 0$, $\forall j$ the m-1 single chain sequences from γ_i to $(j \neq i)$ $[\gamma_i - \gamma_i \cap \chi_j] \cdot \alpha^j_{n_j}$ are feasible. Now consider one of these sequences.

It leads from γ_i to say γ' (say chain j_1 is finished). Then as $d(\gamma_i \cap \chi_{j_1})$ is replaced by 0, the uni-chain sequence down chain j_2 which was applicable to γ_i must be applicable to γ', leading to $\gamma'' \equiv [\gamma_i - \gamma_i \cap \chi_{j_1} - \gamma_i \cap \chi_{j_2}] \cdot \alpha^{j_1}_{n_{j_1}} \cdot \alpha^{j_2}_{n_{j_2}}$.

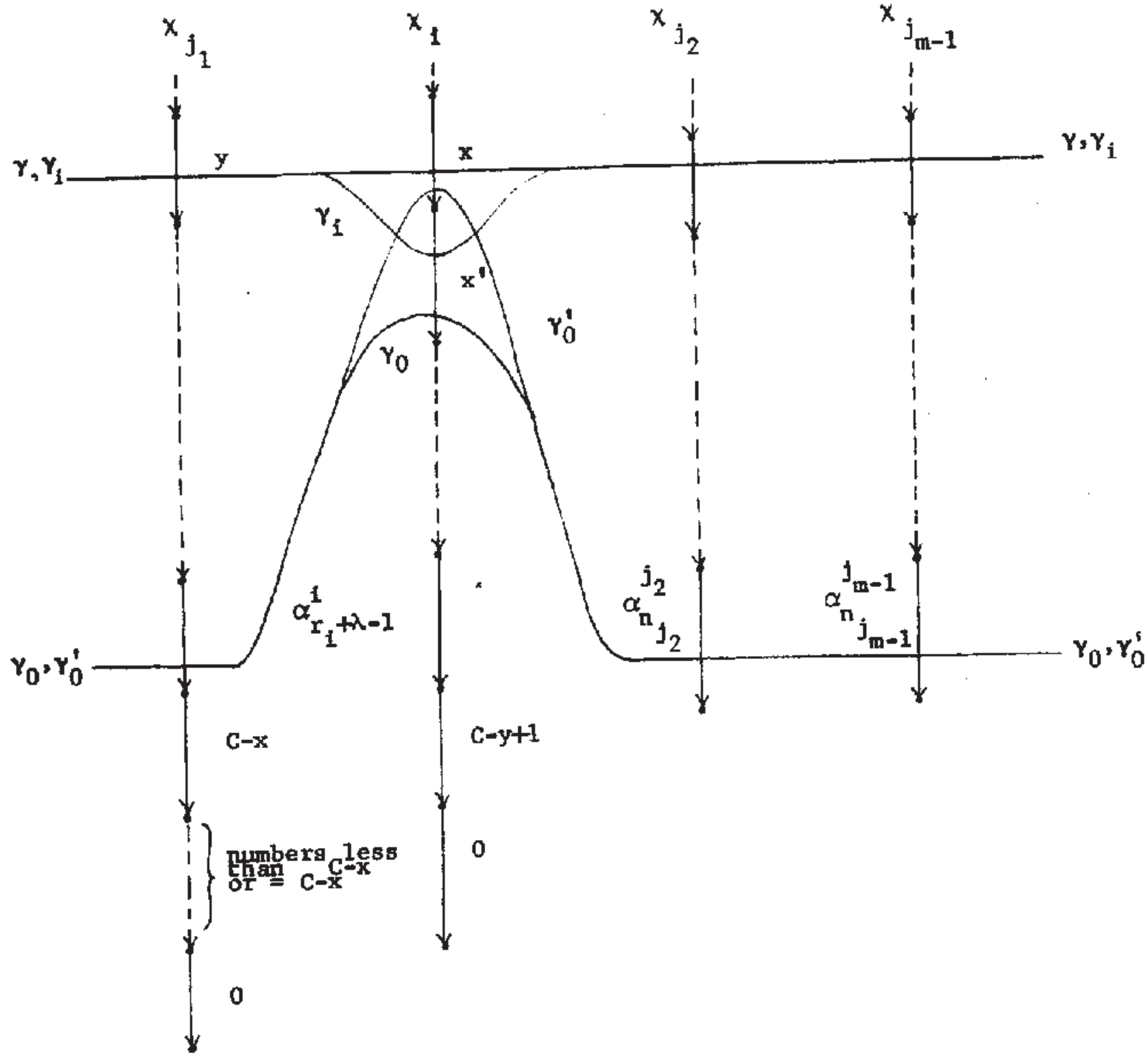
Continuing in this manner it is clear that a feasible sequence exists from γ_i to

$\gamma_0 \equiv \alpha^1_{n_1} \; \alpha^2_{n_2} \cdots \alpha^i_{r_i} \cdots \alpha^m_{n_m}$ where $\alpha^j_{n_j}$ is the terminal arc on χ_j,

Now as $d(\alpha^1_{n_1}) = d(\alpha^2_{n_2}) \cdots d(\alpha^m_{n_m}) = 0_r$, and d (any arc) $\leq$ C it is clear that this sequence must extend to γ_T. As a result γ_i must be safe. This is part (a) of the theorem.

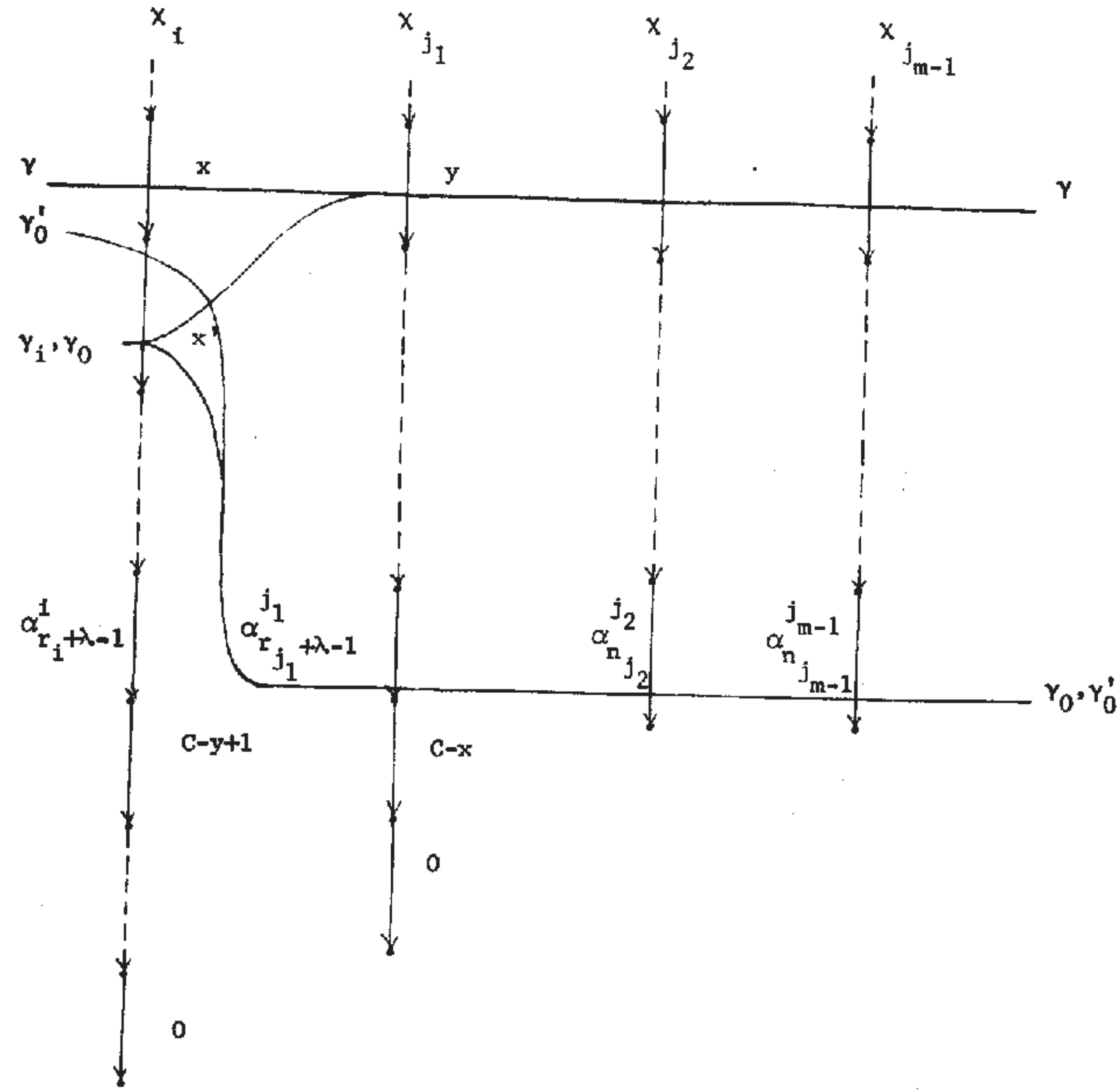
Part (b) of the theorem is proved by showing a general example in which γ_i can be unsafe or safe in spite of the feasibility of all sequences of length λ-1 (the case $k < \lambda-1$ will then be seen to be obvious). By virtue of the definition of λ, none of the k-length sequences can involve arcs below $\alpha^\ell_{r_\ell+k}$ (the arc reached by making all k moves down χ_ℓ) or $\alpha^i_{r_i+k}$ so that their feasibility imposes no constraints on the demands associated with these arcs. Two examples are actually shown, one corresponding to the case $\ell = i$ (figure 4a) and the other to the case $n_i - r_i > n_\ell - r_\ell$ (figure 4b).

In both cases a feasible sequence exists from χ_i to γ_0 as in the proof of part (a) above. However it is obvious that this sequence is not extensible so that γ_i is unsafe. However γ can be safe because



$x' \geq x+1$

Figure 4a



$x' \geq x+1$

Figure 4b

a sequence from γ to $\gamma_0' \equiv [\gamma_0 - \gamma_0 \cap \chi_i] \cdot \gamma \cap \chi_i$ can be extended down chain j_1 to $[\gamma \cap \chi_i] \cdot \alpha^j_{n_j}$ $(\forall j \neq i)$ and then as $d(\alpha^j_{n_j}) = 0$ and $d(\alpha^i_r) \leq$ capacity this sequence can be further extended to γ_T. Thus γ can still be safe. This proves part (b). Q.E.D.

It will be noticed that in the proofs of Theorems 2 and 3 above the safeness of γ was not actually used. It was merely shown that the safeness of γ does not produce compatibility problems. Thus no tests of the (k, ℓ) feasibility type seem to be simplified (shortened) by the safeness of the predecessor state. Secondly, let $n_j' = n_j - r_j$ then the two theorems bound k for (k, ℓ) feasibility tests absolutely by $(n_1' + n_2' + \cdots n_m')-3$ above and by the next to largest of $\{n_1'-1, n_2'-1, \cdots n_m'-1\}$ below. Moreover, the least upper bound for the value of k needed for completeness of a (k, ℓ) feasibility appears to decrease as ℓ is increased ; this is hardly surprising.

In a corollary to Lemma 2 it was indicated that if a feasible path exists up to rank L-m and the last slice has m successors then the sequence can be extended to γ_T. Why then is the bound for a (k, 1) feasibility test L-3 rather than L-m? The reason is that it is not known in a (k, 1) feasibility test whether the last slice has m successors when k = L-m. In fact only one slice at rank L-m has m successors. viz the one corresponding to the decomposition m = 1+1+1+1 ····· m times. There is a special case when the (k,1) test can be stopped at $L - n_k'$ where $n_k' = \max \{n_1', n_2' \cdots n_m'\}$, viz the one where the sequence from γ_1 to $\alpha^1_{n_1} \alpha^2_{n_2} \cdots \alpha^k_{r_k} \cdots \alpha^m_{n_m}$ is feasible (in the last slice only chain χ_k remains). What these two special cases point out is (k, 1) feasibility tests should be modified to detect such special cases in actual use.

Once again it appears that the safeness algorithm specified in [1] is the preferred test as it cannot be worse than the (k, 1) feasibility test.

Reference

[1] Hebalkar, P. G., "Coordinated Sharing of Resources in Asynchronous Systems", Project MAC, Computation Structures Group Memo No. 45, January, 1970.