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Informal Outline of a Theory of Parallel Computation Schemata for Simple Algorithms

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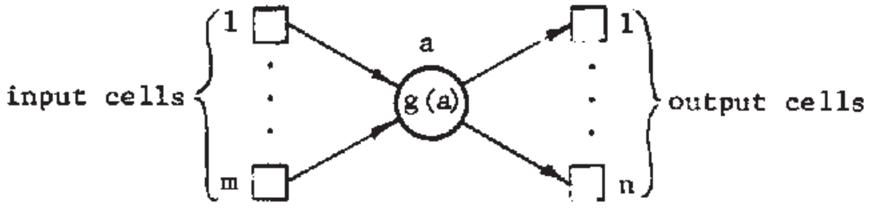
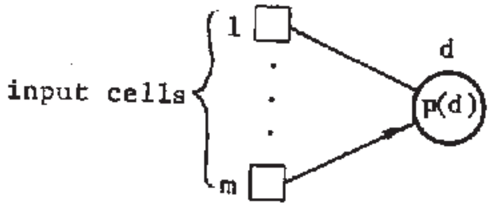
Work on computation schemata in the Computation Structures Group has evolved from the thesis research done by Van Horn [7], Rodriguez [5], Luconi [3], and Slutz [6] at Project MAC, and has been considerably influenced by the original studies of Yanov [8] and, more recently, the work of Karp and Miller [2], and the work of Paterson [4]. Two questions have been of greatest interest to us: What sort of constraints must be met in representation of parallel computations so that unique results of computations may be guaranteed? Under what conditions is it possible to determine whether two representations (schemata) describe identical classes of computations? For the class of schemata we have considered, we have satisfactory answers to the first question, and have gained a much better understanding of the second.

A computation schema represents the manner in which functional elements and decision elements are interconnected, and their action sequenced, to define an algorithm. The functional elements of a schema are called <u>operators</u>: Each operator a evaluates some unspecified function of an m-tuple of input variables and assigns values to an n-tuple of output variables.



$g(a)$: {m-tuples} → {n-tuples}

The unspecified function associated with an operator a is denoted by g(a). The decision elements of a schema are called <u>deciders</u>: Each decider d tests some unspecified predicate p(d) for an m-tuple of input variables



p(d): {m-tuples} → {<u>true</u>, <u>false</u>}

A computation schema has two parts — a <u>data flow graph</u> and a <u>control</u>. The data flow graph defines the interconnections through which results of each operator application are passed on as arguments for further transformations and tests. The variables of a schema are represented in the data flow graph by boxes called <u>cells</u>. There is also a circle for each operator and a diamond for each decider. Directed arcs join the operators to their output cells and represent the connections to each operator and decider from its input cells.

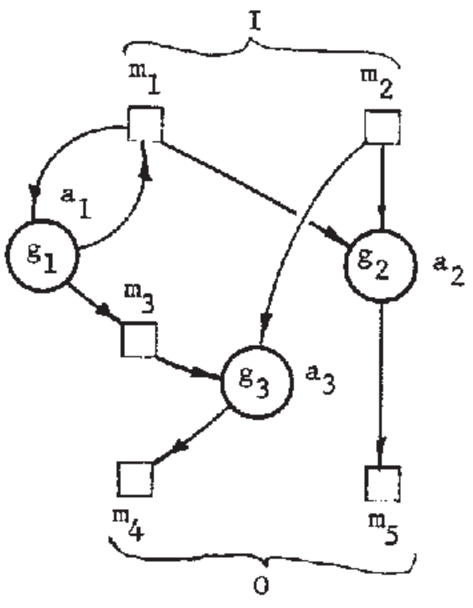
A data flow graph

input cells

$I = \{m_1, m_2\}$

output cells

$O = \{m_4, m_5\}$



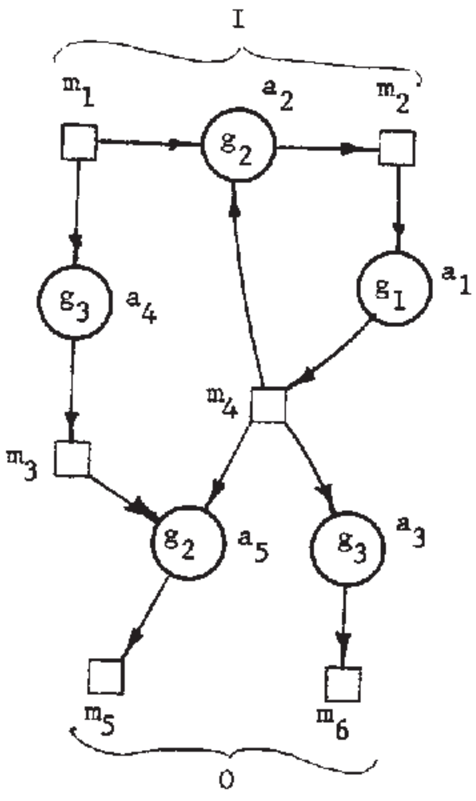
The cells of the schema are identified by the letters m_1, m_2, ... Certain cells are designated as input or output cells. Values are assigned to the input cells before a computation begins; upon completion, the result is the set of values present in the output cells. Several operators, say a and b, may have the same associated function letter: $g(a) = g(b)$. In this way, a schema may require that two operators, a and b, always implement the same transformation, although the particular transformation is unspecified. Similarly, each decider d designates a predicate letter $p(d)$. The function letters and predicate letters of a schema make up two finite sets G and P.

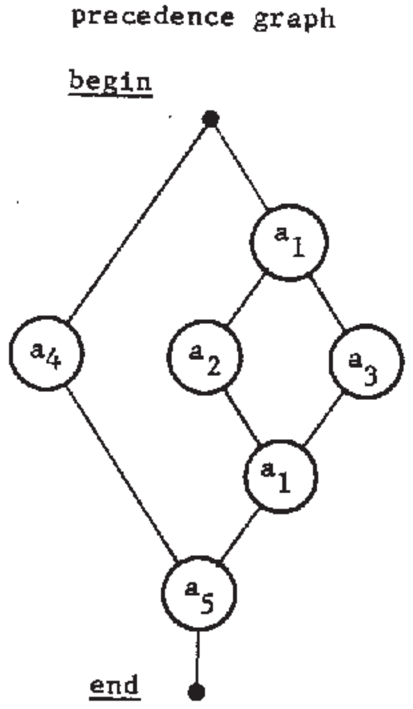
The control of a computation schema is a specification of the sequences in which operators and deciders are permitted to act. In particular, the control will indicate how further progress of computations by the schema is affected by the results of actions by the deciders. A variety of different representations have been used for the control in the literature on the subject. We shall begin by considering properties of the set of action sequences permitted by the schema. Let A be the set of operators. For each decider d_i, let t_i stand for an action by the decider for which the outcome is <u>true</u>; let f_i stand for an action by decider d_i for which the outcome is <u>false</u>. Let $T = \{t_1, t_2, \ldots\}$, $F = \{f_1, f_2, \ldots\}$. Then any string of letters in the set

$$V^* \quad \text{where} \quad V = (A \cup T \cup F)$$

might be a <u>control sequence</u> of the schema. By the <u>control set</u> of the schema, we mean the set of those sequences in V^* that are permitted by the control. Before further characterizing properties of control sets let us consider a few examples of schemata. For these examples we shall represent the control through the use of precedence graphs [1].

S_1: data flow graph



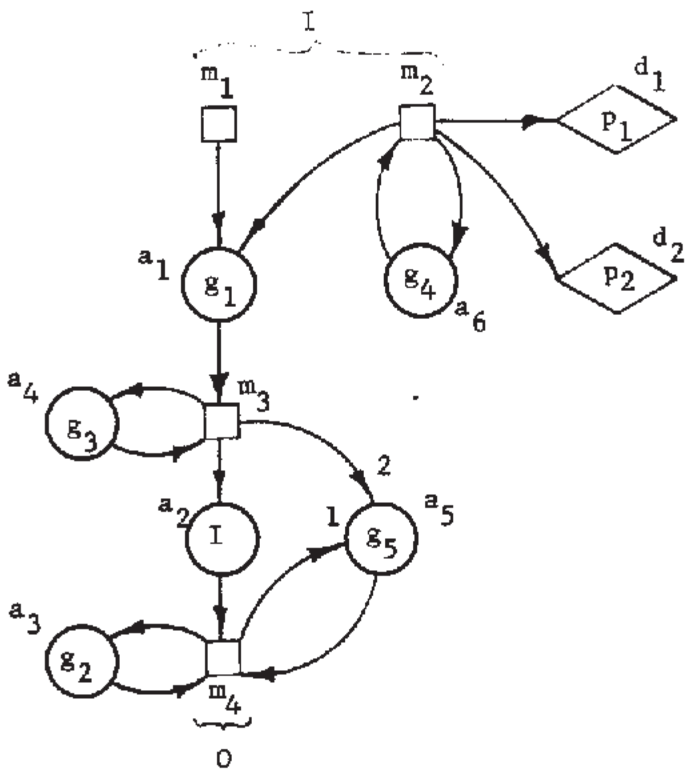


This is an elementary schema because there are no deciders in the data flow graph. A sequence in the control set of S_1 must contain each node (operator instance) in the precedence graph in an order consistent with the partial ordering indicated by the graph.

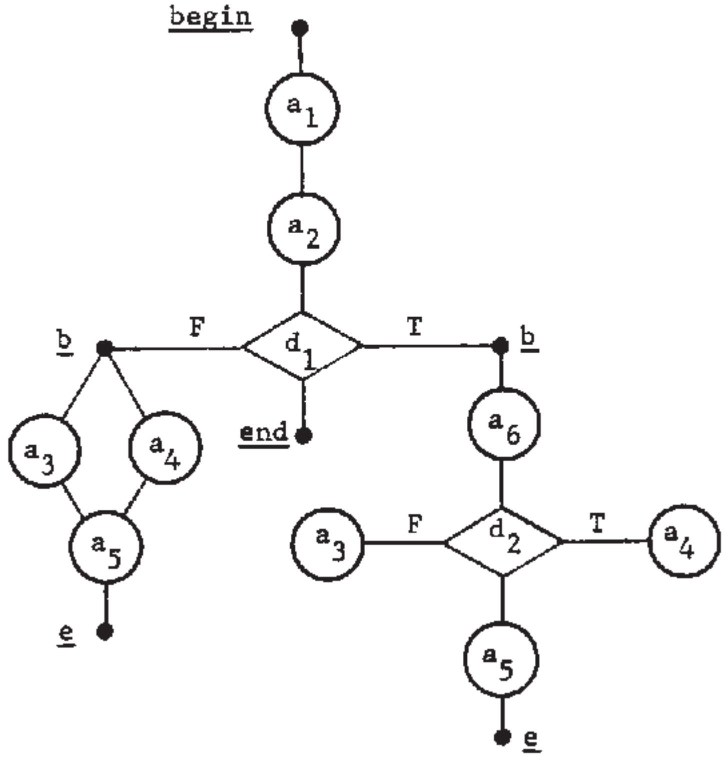
$$C_1: \left\{\begin{array}{ll} (a_1\ a_2\ a_3\ a_1\ a_4\ a_5) & (a_1\ a_3\ a_4\ a_2\ a_1\ a_5) \\ (a_1\ a_3\ a_2\ a_1\ a_4\ a_5) & (a_1\ a_4\ a_2\ a_3\ a_1\ a_5) \\ (a_1\ a_2\ a_3\ a_4\ a_1\ a_5) & (a_1\ a_4\ a_3\ a_2\ a_1\ a_5) \\ (a_1\ a_3\ a_2\ a_4\ a_1\ a_5) & (a_4\ a_1\ a_2\ a_3\ a_1\ a_5) \\ (a_1\ a_2\ a_4\ a_3\ a_1\ a_5) & (a_4\ a_1\ a_3\ a_2\ a_1\ a_5) \end{array}\right\}$$

When deciders are present, it is no longer true that all sequences in the control set are of the same length.

S_2: data flow graph



precedence graph



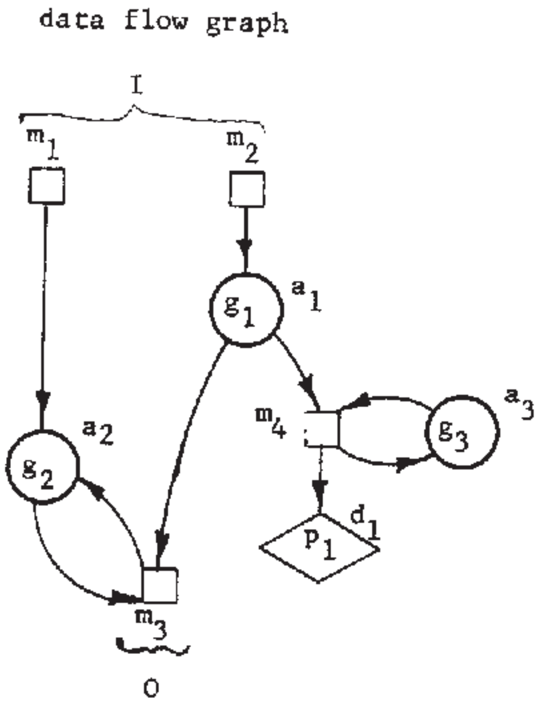
The diamond nodes in the precedence graph connect to two subordinate precedence graphs that specify alternative computations according as the designated decider has a true or false outcome. Operator a_2 in the data flow graph is an identity operator; the associated function $f(a_2)$ is always the identity function.

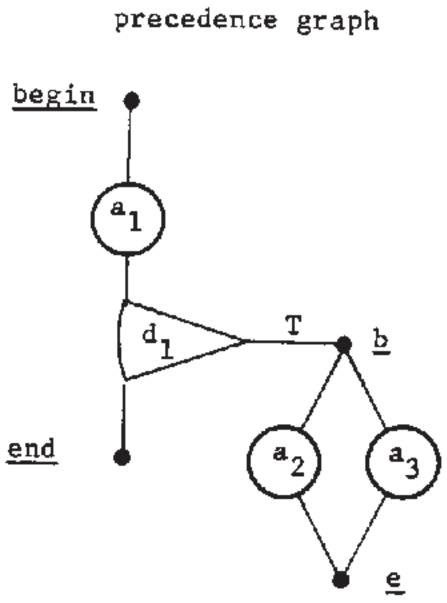
$$C_2: \left\{ \begin{array}{l} (a_1\ a_2\ f_1\ a_3\ a_4\ a_5) \\ (a_1\ a_2\ f_1\ a_4\ a_3\ a_5) \\ (a_1\ a_2\ t_1\ a_6\ f_2\ a_3\ a_5) \\ (a_1\ a_2\ t_1\ a_6\ t_2\ a_4\ a_5) \end{array} \right\}$$

Since no iteration is present, the control set C_2 is finite.

Iteration is indicated in a precedence graph by a pie-shaped node connected to a subordinate precedence graph.

S_3:





The computation specified by the subgraph is repeated until the decider acts with a false outcome. The control set is C_3.

$$C_3: \left\{ \begin{array}{l} (a_1\ f_1) \\ (a_1\ t_1\ a_2\ a_3\ f_1) \\ (a_1\ t_1\ a_3\ a_2\ f_1) \\ (a_1\ t_1\ a_2\ a_3\ t_1\ a_2\ a_3\ f_1) \\ \quad\cdot \qquad\quad \cdot \\ \quad\cdot \qquad\quad \cdot \\ \quad\cdot \qquad\quad \cdot \end{array} \right\}$$

$$C_3 = a_1\ [t_1(a_2\ a_3 \cup a_3\ a_2)]^*\ f_1$$

To convert a computation schema into a specification of a particular algorithm it is necessary to specify the functions and predicates designated by the function letters in G and the predicate letters in P. Of course, the specified functions and predicates must have domains and ranges consistent with the topology of the data flow graph, and in agreement wherever the value of a function may be the argument of a function or predicate. Such a specification of functions and predicates is called (after Yanov) an <u>interpretation</u> of a schema.

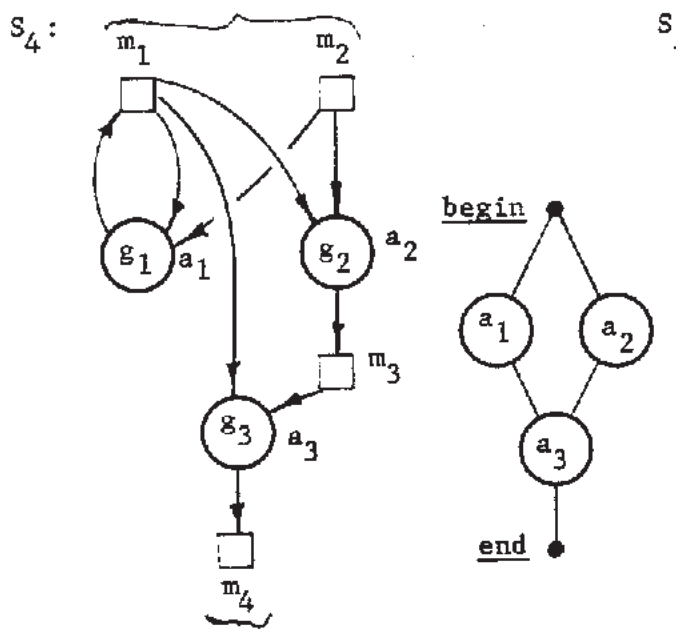
Two properties of schemata are of particular interest to us. A schema S is <u>determinate</u>* if, for any interpretation of the function and prediate letters, S determines a functional relation of output tuples to input tuples. In order to say whether two schemata S_1 and S_2 describe the same computations, we must be able to relate the interpretations of the function and predicate letters in S_1 and S_2. For this purpose let

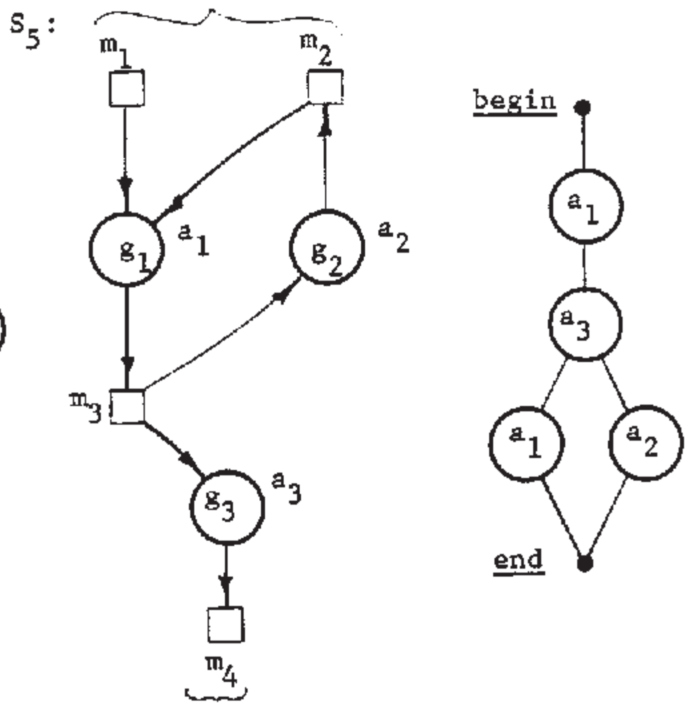
$$G = G_1 \cup G_2 \qquad P = P_1 \cup P_2$$

Then S_1 and S_2 are <u>equivalent</u>* schemata if, for any interpretation of the functions and predicate letters in G and P, S_1 and S_2 determine precisely the same relation of output tuples to input tuples. (Note that equivalence applies to nondeterminate schemata as well as to determinate schemata.)

The following schemata illustrate the definition of determinacy.

*These definitions are weaker than those used by Karp and Miller in [2], hence our results on determinacy and equivalence are stronger than would follow by application of their work.





Schema S_4 is not determinate because the sequences

$$\alpha = (a_1\ a_2\ a_3) \qquad \beta = (a_2\ a_1\ a_3)$$

are both in C_4 but correspond to different compositions of functions

$$\alpha:\ m_4 = g_3(g_1(m_1, m_2), g_2(g_1(m_1, m_2), m_2))$$

$$\beta:\ m_4 = g_3(g_1(m_3, m_2), g_2(m_1, m_2))$$

It is easy to choose functions to associate with the function letters so the functions defined by these compositions are distinct. Schema S_5 is determinate even though the sequences

$$\alpha = (a_1\ a_3\ a_1\ a_2) \qquad \beta = (a_1\ a_3\ a_2\ a_1)$$

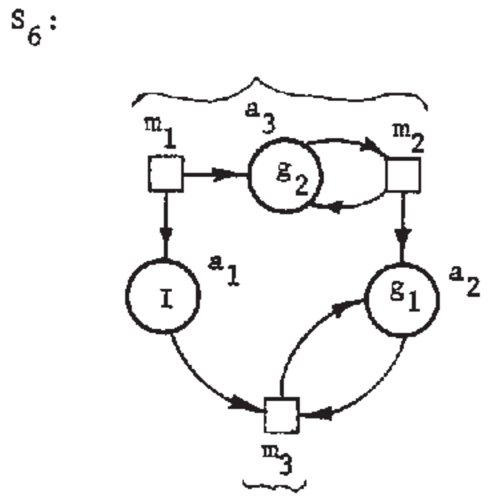
leave different values in cell m_3. This is because the final value in cell m_3 does not participate in determining the final value put in the output cell m_4. Yet sequences α and β determine the same composition of functions as the relation of output values to input values.

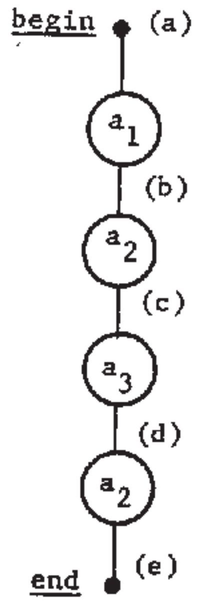
$$\begin{aligned} \alpha:\quad m_4 &= g_3(g_1(m_1, m_2)) \\ m_2 &= g_2(g_1(m_1, m_2)) \\ m_3 &= g_1(m_1, m_2) \end{aligned}$$

$$\begin{aligned} \beta:\quad m_4 &= g_3(g_1(m_1, m_2)) \\ m_2 &= g_2(g_1(m_1, m_2)) \\ m_3 &= g_1(m_1, g_2(g_1(m_1, m_2))) \end{aligned}$$

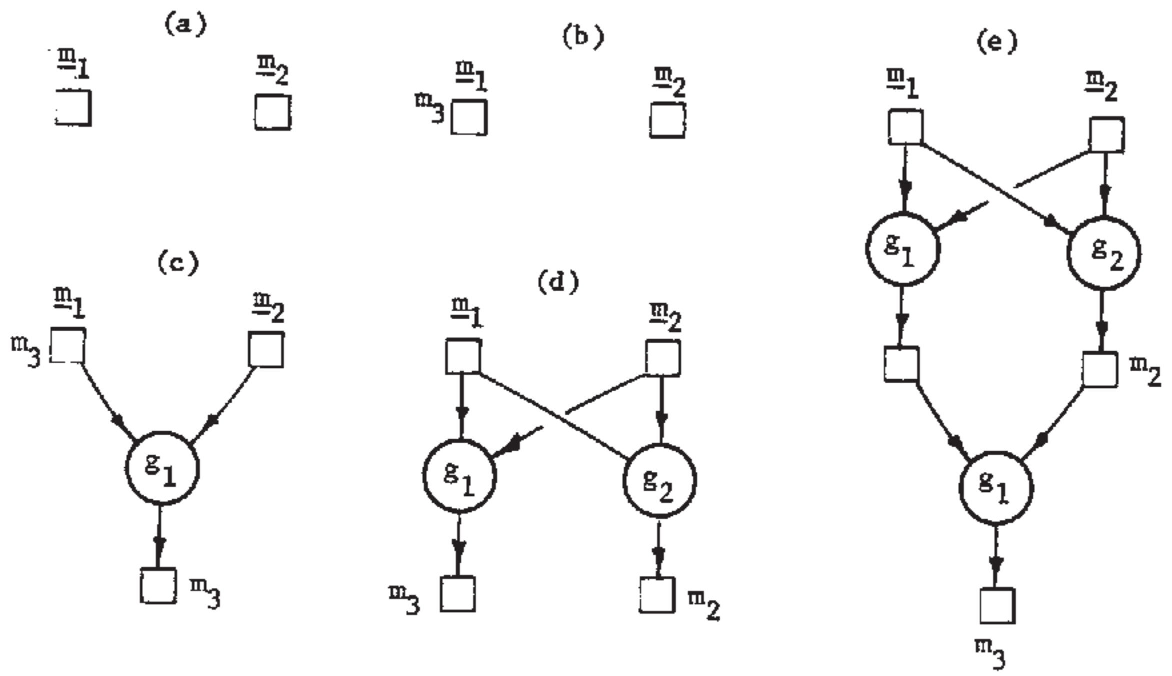
To develop insight into the question of determinism and equivalence we have devised the notion of data-dependence graph or dadep graph for short. A dadep graph of a schema sets forth separately each action by an operator or decider. For a particular control sequence of a schema the final value placed in each output cell will be the result of some cascaded composition of functions. A dadep graph is just a graph representation of the cascade composition associated with each output cell.

Let us construct the dadep graph for schema S_6 from its unique control sequence $\alpha = (a_1\ a_2\ a_3\ a_4)$.



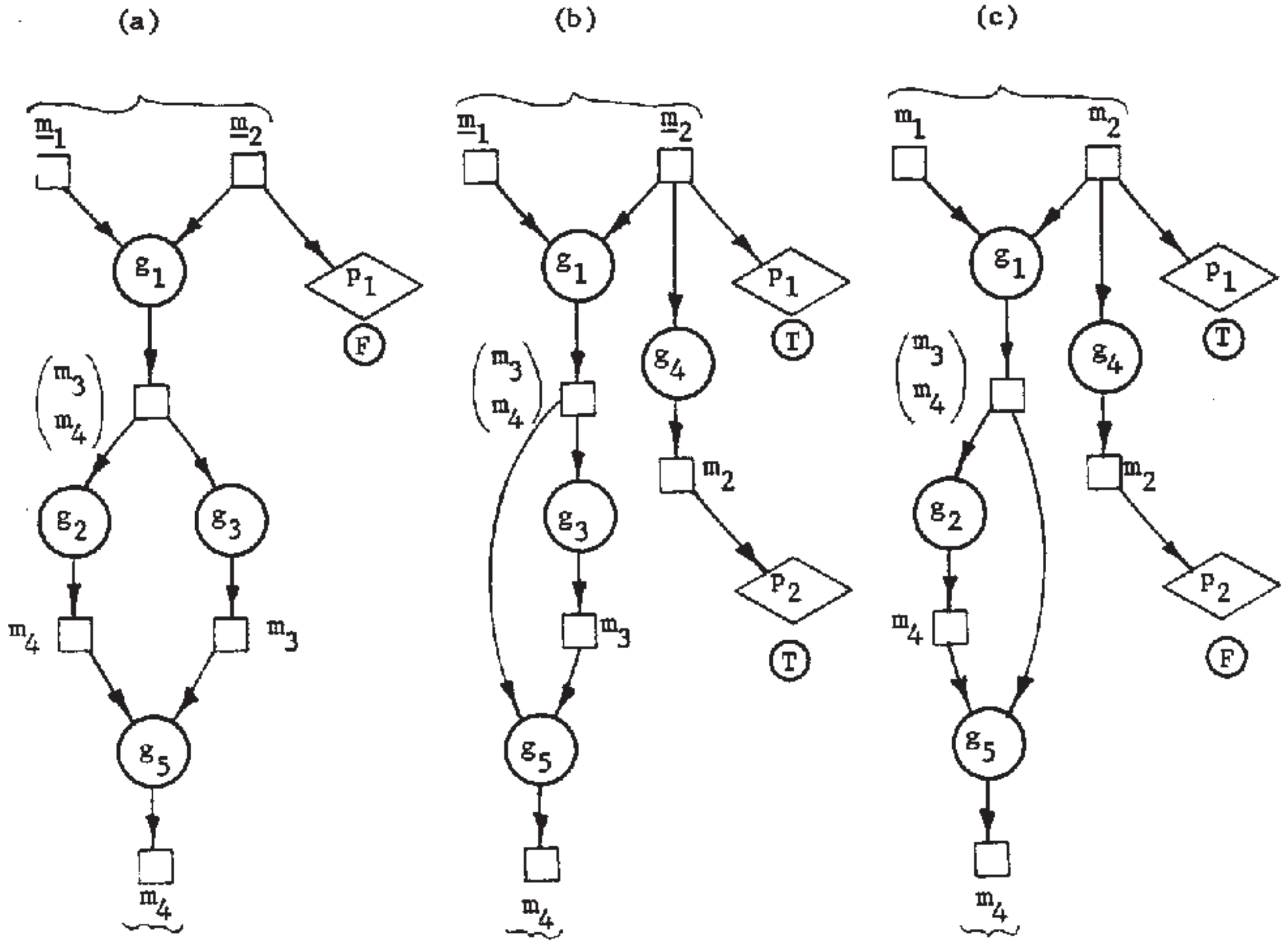


We start by setting down a copy of each input cell of the schema. (The letters denoting input cells are underlined.) Then we add a copy of an operator and its output cells for each succeeding element of the control sequence:

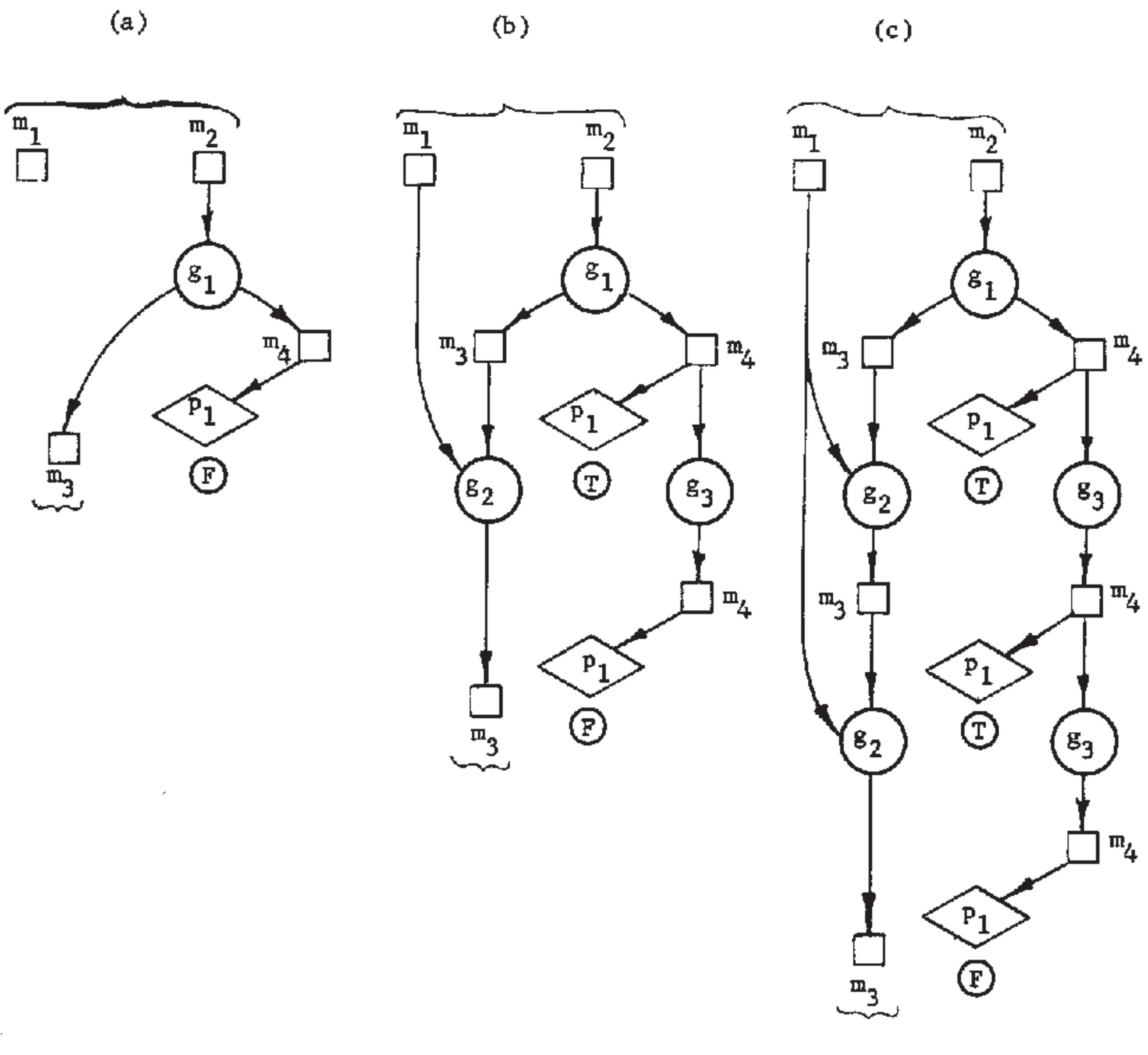


Each cell added to the dadep graph is labelled as in the data flow graph and this label is erased from the cell copy previously bearing it. In the case of an identity operator, a second label is given to the latest copy of its input cell, and no copy of the operator is made.

For schemata that include deciders, there will be a cascade composition of functions associated with each action of a decider as well as each output cell. A determinate schema with k deciders could have 2^k distinct dadep graphs -- one for each combination of decisions that might occur in the course of some computation. For the schema S_2 there are just three dadep graphs because a decision of false by d_1 results in the absence of any action by d_2.



In general a schema that represents an iteration defines an infinite set of dadep graphs. In the case of S_3 the three simplest dadep graphs are:



Nondeterminate computation is represented by a schema when there is a cell that could be assigned a value by one operator either before or after a value is assigned to or read from the cell by another operator or decider action. When this can happen we say the schema has conflict.

Let $O_{(a)}$ denote the set of output cells of an operator a, and let I(x) denote the set of input cells of x which may be an operator or a decider. Then there is conflict between a and x if there are sequences

$$\overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ a\ \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\varphi} \quad \text{and} \quad \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ x\ \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\psi}$$

in the control set of a schema such that

$$O(a) \cap I(x) \neq \emptyset$$

There is conflict in both schemata S_4 and S_5, as demonstrated by these pairs of control sequences:

$$\begin{tikzcd}[row sep=small, column sep=tiny]
S_4: & (a_1 \arrow[d, leftrightarrow] & a_2 & a_3) & & S_5: & (a_1 & a_3 & a_1 \arrow[d, leftrightarrow] & a_2) \\
& (a_2 & a_1 & a_3) & & & (a_1 & a_3 & a_2 & a_1)
\end{tikzcd}$$

$$O(a_1) \cap I(a_2) = \{m_1\} \qquad\qquad O(a_1) \cap I(a_2) = \{m_3\}$$
$$O(a_2) \cap I(a_1) = \{m_2\}$$

Schemata whose control sets can be represented by precedence graphs share two important properties: Whenever there is the possibility of two actions proceeding concurrently, the occurrence of one of the actions does not inhibit or block the other. This property is called <u>persistence</u>. Expressed as a property of the control set, persistence requires that, if

$$\overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ x \quad \text{and} \quad \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ y$$

are prefixes of some control sequences, then

$$\overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ xy \quad \text{and} \quad \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ yx$$

are also prefixes of some control sequences. The second property is this: If either of two actions may occur before the other, then the order in which they occur has no effect on the subsequent course of the computation. In terms of control sequences

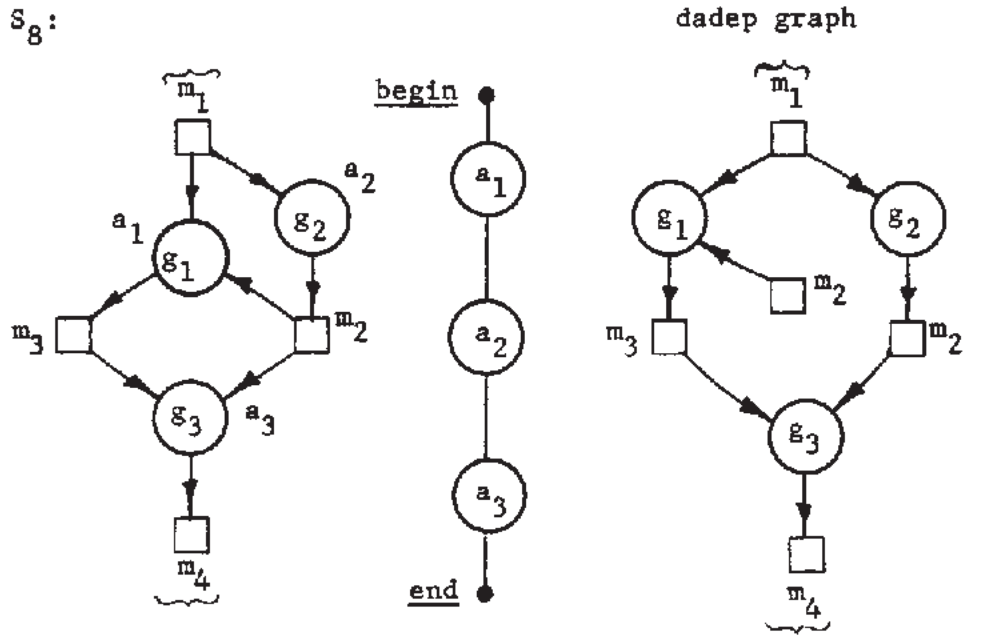
$$\begin{matrix} \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ xy\ \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\varphi} \in C \\ \underbrace{\cdot\ \cdot\ \cdot\ \cdot}_{\omega}\ yx\ \underbrace{\cdot\ \cdot\ \cdot\ \cdot}_{\psi} \in C \end{matrix} \quad \text{imimplies} \quad \begin{matrix} \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\omega}\ yx\ \overbrace{\cdot\ \cdot\ \cdot\ \cdot}^{\varphi} \in C \\ \underbrace{\cdot\ \cdot\ \cdot\ \cdot}_{\omega}\ xy\ \underbrace{\cdot\ \cdot\ \cdot\ \cdot}_{\psi} \in C \end{matrix}$$

A schema satisfying this condition on its control set is said to be <u>commutative</u>. Any schema (as in the examples) with a control that can be represented by a precedence graph is both persistent and commutative. (The converse is not true, however.)

By an adaptation of known methods it is not difficult to show that any computation schema that is persistent, commutative, and free of conflict is guaranteed to be determinate. A more interesting problem is to determine the circumstances for which the conflict-free property is a necessary condition for schemata to be determinate. We have found two natural restrictions on schemata sufficient that any determinate (and persistent, commutative) schema is necessarily conflict free. The first of these restrictions amounts to requiring that each action by any operator or decider in a schema participate in determining some output value. A schema meeting this restriction is said to be <u>normal</u>. The second restriction disallows control sets that permit repetition of a computation or test for the same m-tuple of input values.

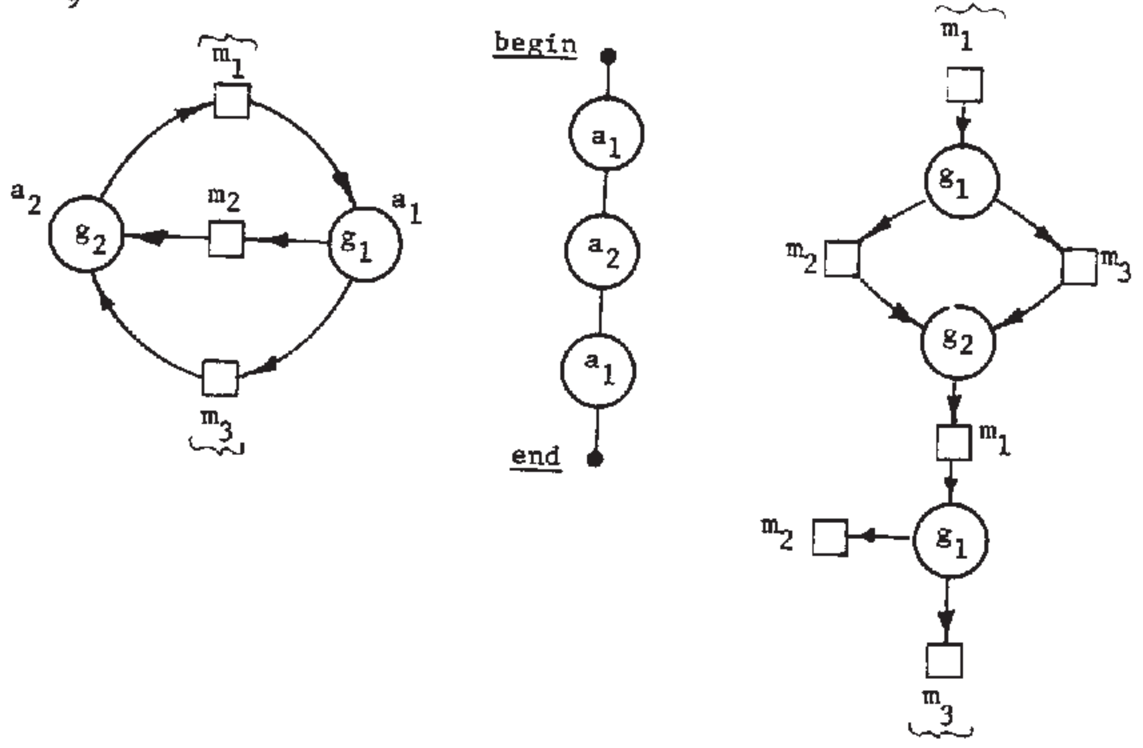
A schema meeting this restriction is said to be repetition free.

To be normal, a schema must first be well defined. That is, in any situation that an operator or decider may act, all of its input variables must have been assigned values. The properties well-defined and normal are readily explained in terms of dadep graphs. For a schema to be well-defined, it must have no dadep graph in which some cell that is not a copy of an input cell is not written by some operator. Schema S_8 is not well-defined because there is a copy of cell m_2 in the dadep graph that is read by not written, and m_2 is not an input cell of the schema.

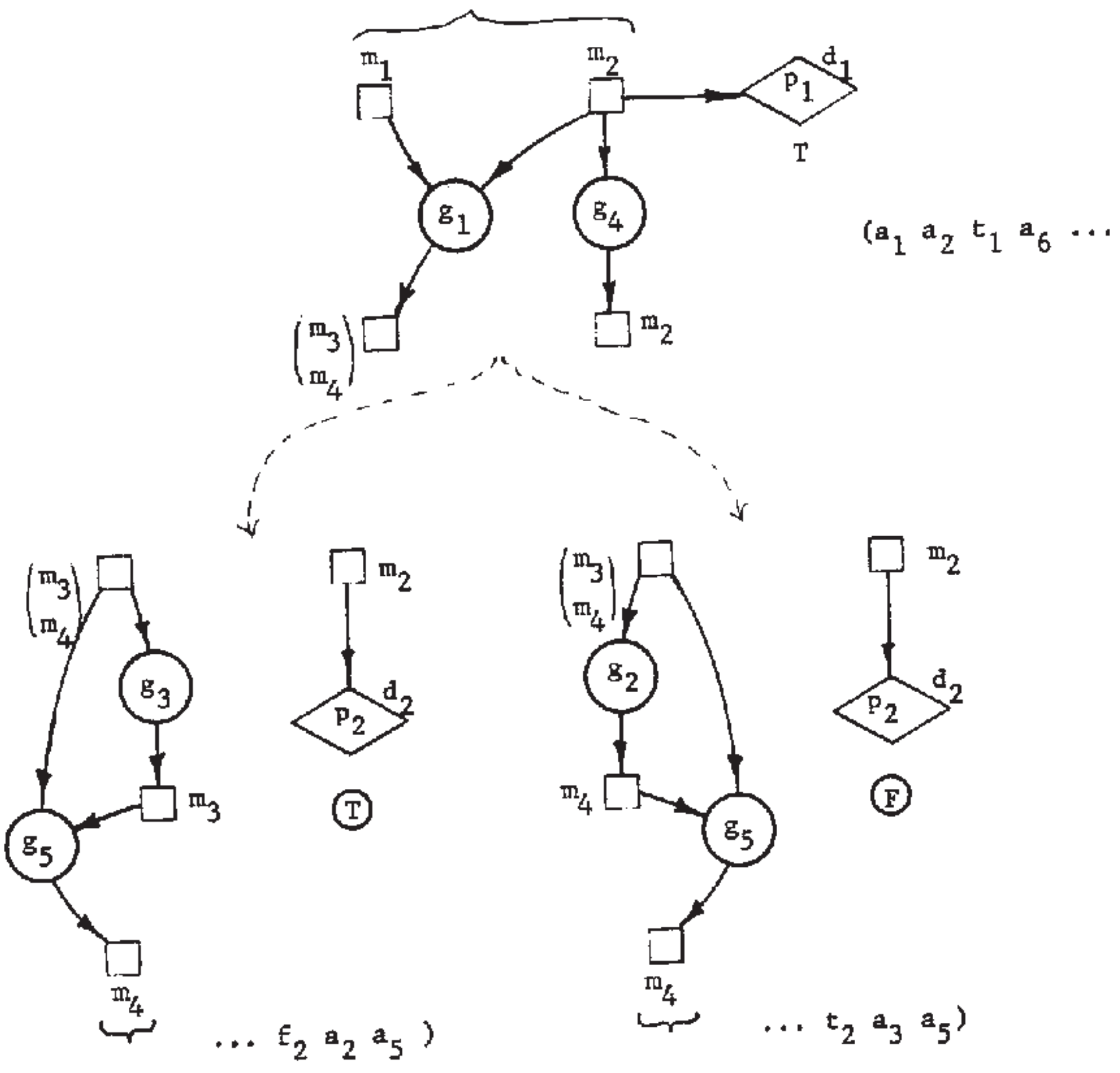


For an elementary schema to be normal, each cell in any dadep graph that is a copy of an input cell, or is written by a copy of an operator must be connected by a directed path through the dadep graph to the final copy of some output cell. Schema S_9 is not normal unless cell m_2 is considered an output cell.

S_9:

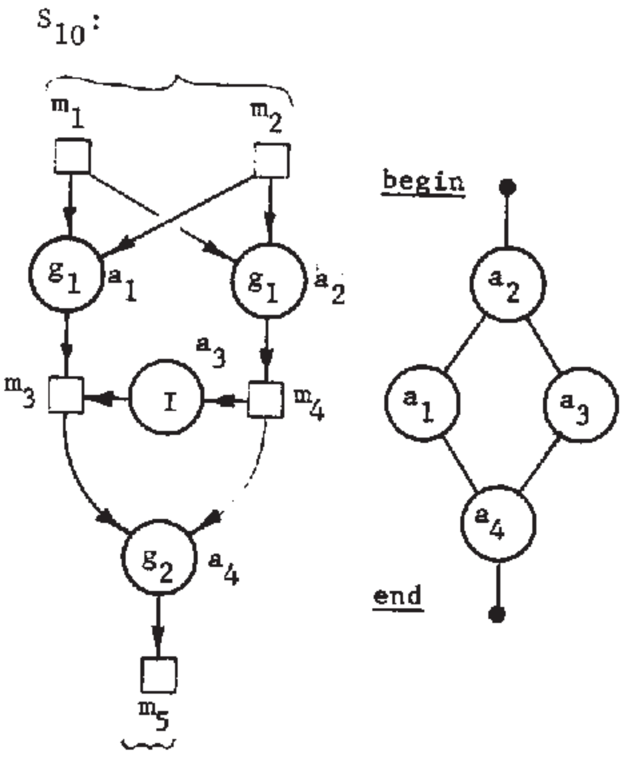


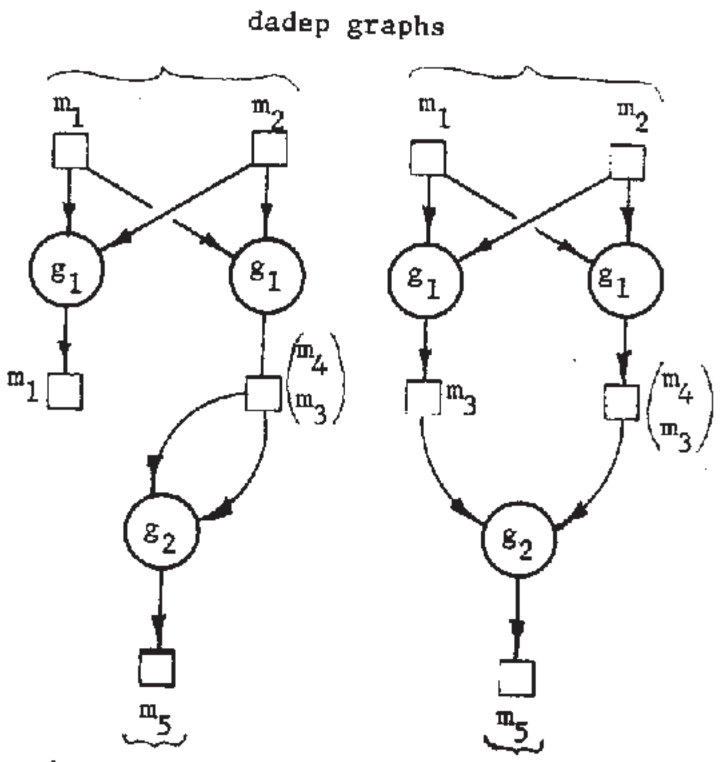
For a schema with deciders to be normal, each input cell to a copy of a decider in any dadep graph must be considered as a final copy of an output cell in applying the rule given above. Furthermore, each decision must have some effect on the future course of the computation. Consider a partial dadep graph for S_2 that is constructed from a prefix of a control sequence such that the next action is a test by decider d_2:



There must be at least two distinct ways of completing the dadep graph, of which one is associated with a true outcome of the test and the other is associated with a false outcome.

Schemata S_1, S_2 and S_3 are all normal. They are also determinate and conflict-free. Nevertheless, it is possible for a normal schema to have conflict and still be determinate.





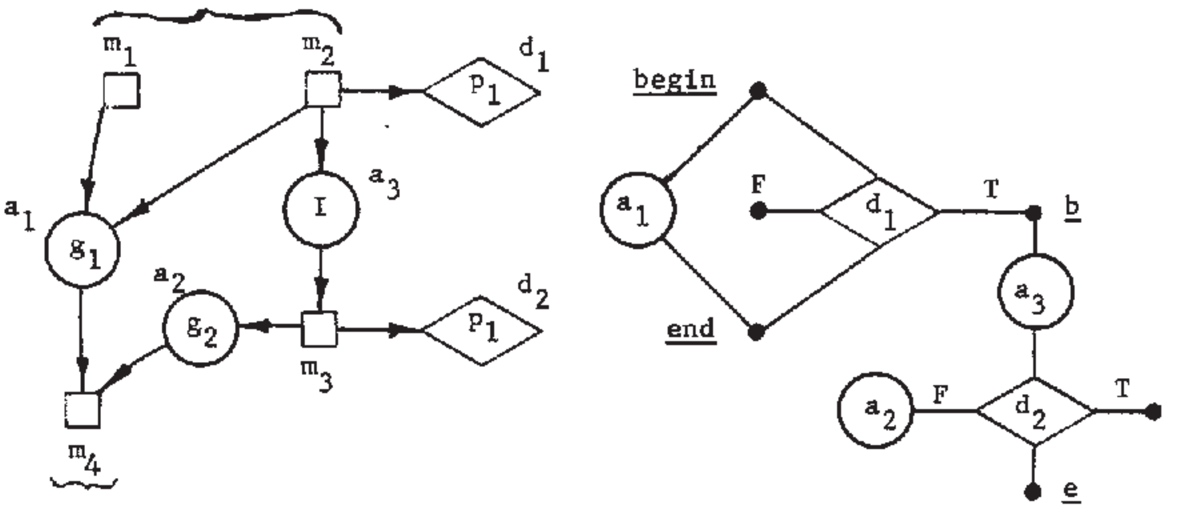
$(a_2\ a_1\ a_3\ a_4)$ $(a_2\ a_3\ a_1\ a_4)$

Schema S_{10} generates two distinct dadep graphs corresponding to the two orders in which a_1 and a_3 may act. Yet the two dadep graphs define the same composition of functions.

$$g_2(g_1(x_1\ x_2),\ g_1(x_1,\ x_2))$$

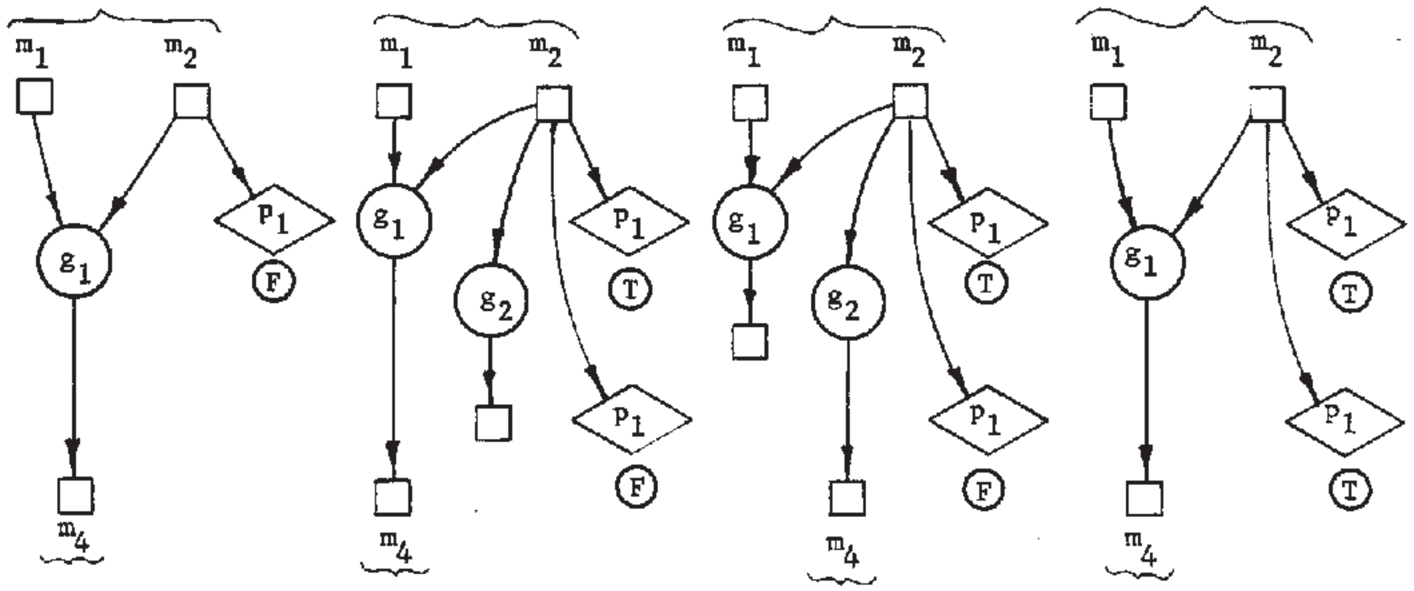
This is possible because the function designated g is applied twice to the same input values. This is known as an <u>operator repetition</u>. A schema may also exhibit a <u>decision repetition</u>.

S_{11}:



Schema S_{11} has these dadep graphs:

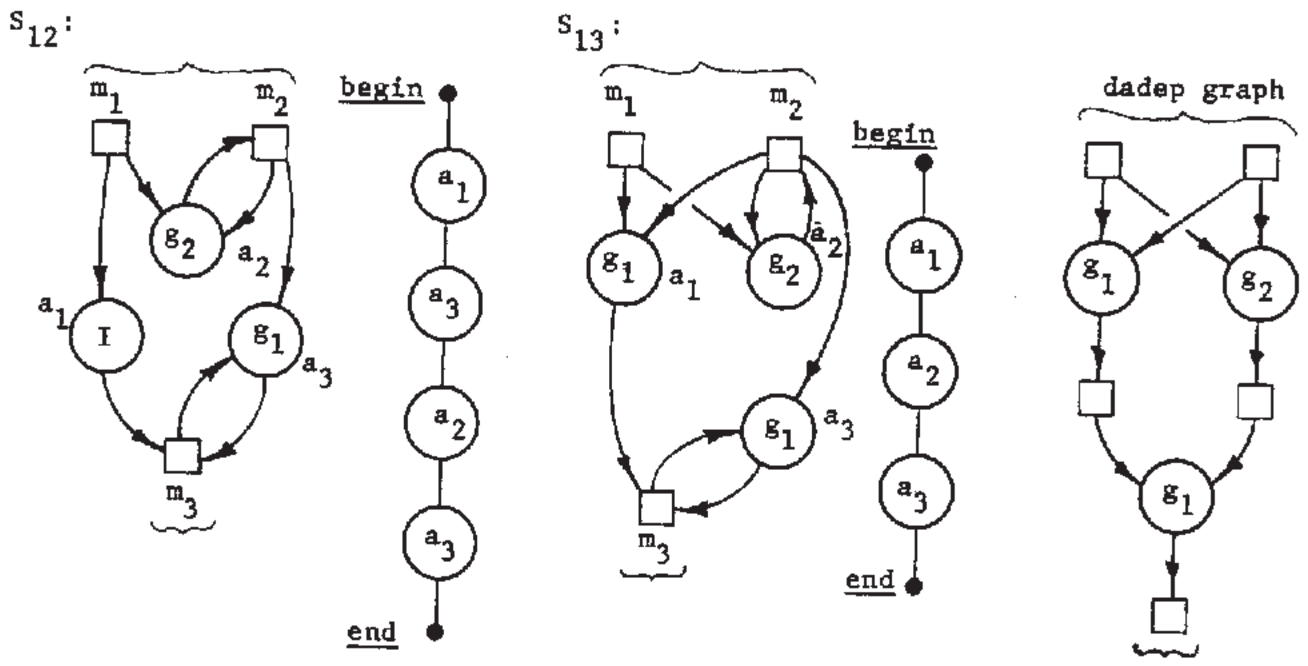
(a) (b) (c) (d)



There is conflict between operators a_1 and a_2 at cell m_4; however, the two dadep graphs (b and c) generated by the conflict cannot represent any computations because the repeated tests of p_1 must yield identical outcomes.

It is evident that repetitions are not especially useful in representing algorithms. Hence we feel justified in restricting our attention to repetition-free schemata. We have been able to show that in any schema that is persistent, commutative, normal and repetition free, the conflict-free property is necessary for the schema to be determinate.

For an elementary schema that is well defined, normal, repetition-free and determinate, all execution sequences yield the same dadep graph. In fact the dadep graph is a canonical form for this class of schemata. Thus the equivalence of any two elementary schemata can be tested by constructing their dadep graphs.



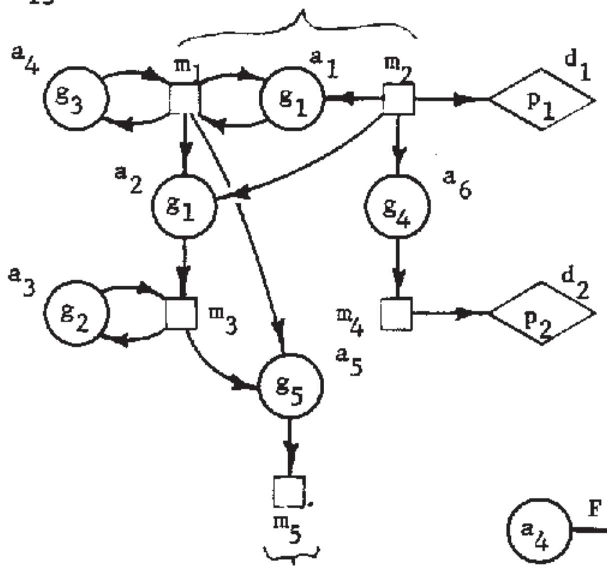
In the case of a normal, repetition free schema that has deciders but no iteration, the class of computations represented is described by a finite set of dadep graphs, as shown for the schema S_2 earlier. Each pair of input values will be processed as shown in that one of the dadep graphs for which the evaluation of predicates agrees with the truth values given at decision points of the graph.

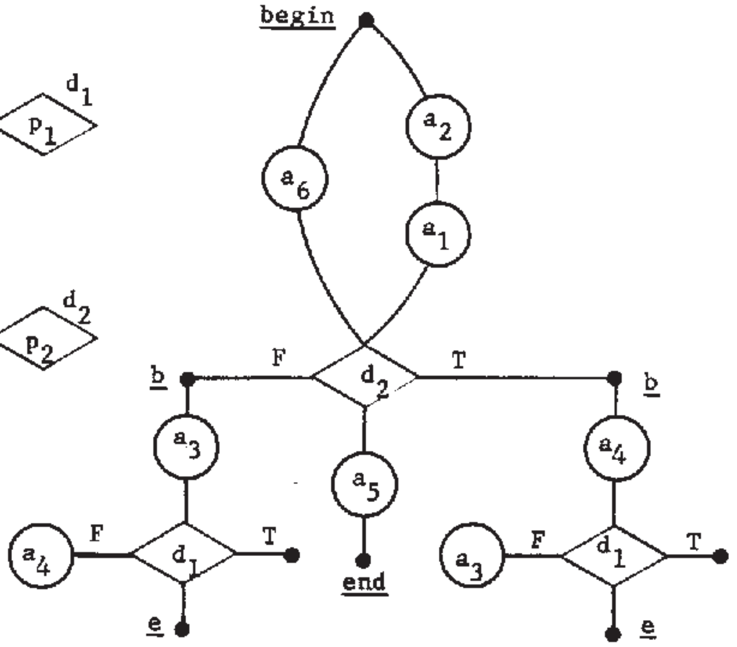
We can construct a table of two columns, called a <u>conditional</u> <u>expression list</u>, that characterizes the computations represented by a schema. Each row of the table corresponds to one dadep graph. In the left-hand column we write a disjunction of the predicates that must be satisfied by the input variables for the corresponding dadep graph to describe the computation. In the right-hand column we write the compositions of functions that specify the corresponding dependence of output values on input values. For S_2 we have:

Condition	Expression
$\bar{p}_1(x_2)$	$g_5(g_2(g_1(x_1, x_2)), g_3(g_1(x_1, x_2)))$
$p_1(x_2) \cdot \bar{p}_2(g_4(x_2))$	$g_5(g_1(x_1, x_2), g_3(g_1(x_1, x_2)))$
$p_1(x_2) \cdot p_2(g_4(x_2))$	$g_5(g_2(g_1(x_1, x_2)), g_1(x_1, x_2))$

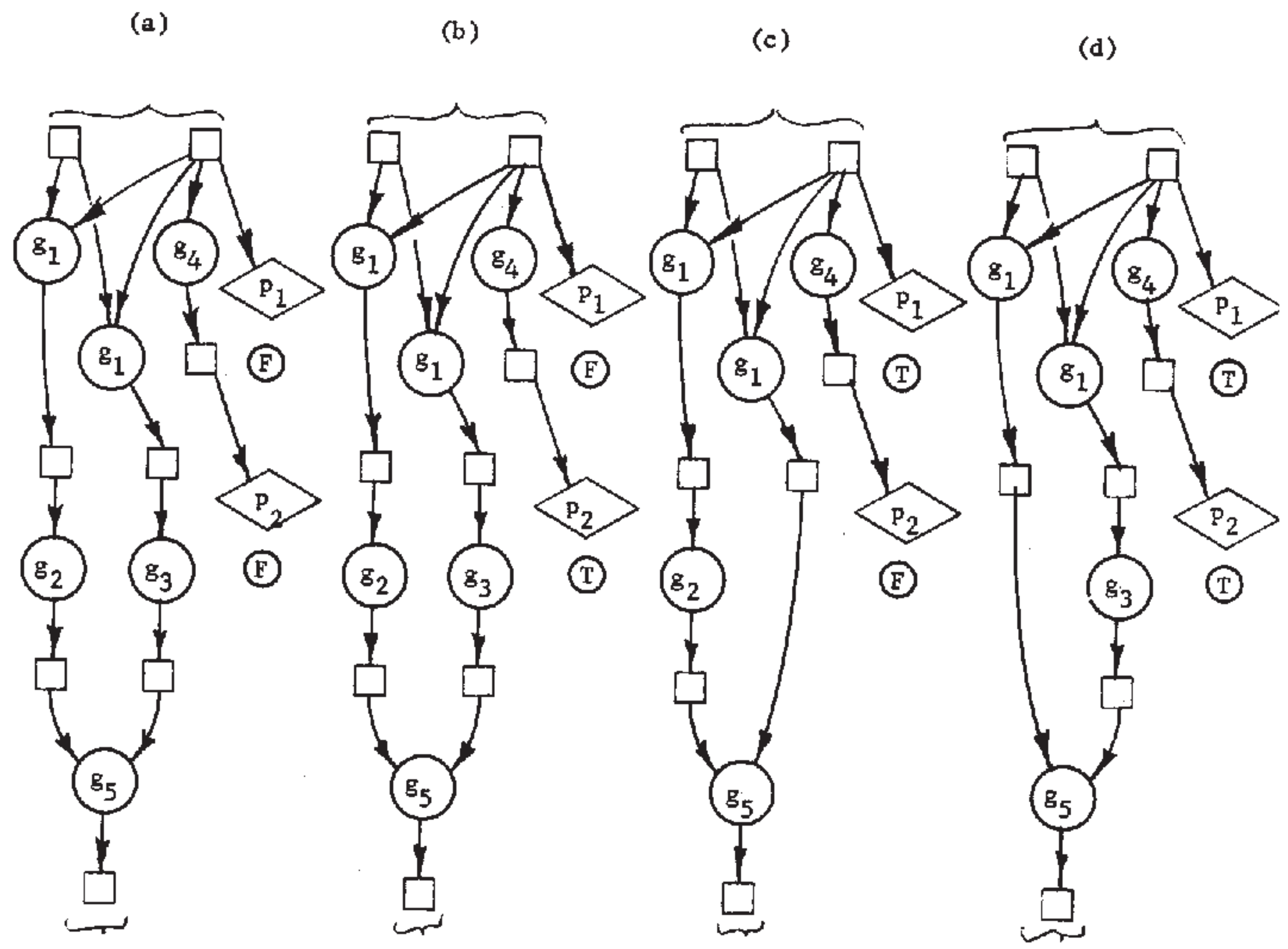
Now consider the schema

S_{13}:





The dadep graphs of S_{13} are



and the corresponding conditional expression list is:

Condition	Expression
$\overline{p}_1(x_2) \cdot \overline{p}_2(g_4(x_2))$	$g_5(g_2(g_1(x_1, x_2)), g_3(g_1(x_1, x_2)))$
$\overline{p}_1(x_2) \cdot p_2(g_4(x_2))$	$g_5(g_2(g_1(x_1, x_2)), g_3(g_1(x_1, x_2)))$
$p_1(x_2) \cdot \overline{p}_2(g_4(x_2))$	$g_5(g_2(g_1(x_1, x_2)), g_1(x_1, x_2))$
$p_1(x_2) \cdot p_2(g_4(x_2))$	$g_5(g_1(x_1, x_2), g_3(g_1(x_1, x_2)))$

This table specifies the same class of computations as does Table 2, for we have the logical equivalence

$$\overline{p}_1(x_2) \equiv \overline{p}_1(x_2) \cdot \overline{p}_2(g_4(x_2)) + \overline{p}_1(x_2) \cdot p_2(g_4(x_2))$$

Thus schemata S_2 and S_{13} are equivalent. In general, noniterative schemata may be tested for equivalence by constructing their conditional expression lists.

Since an iterative schema has an infinite set of dadep graphs, its conditional expression list is infinitely long. For schema S_3 we have

Condition	Expression
$\overline{p}_1(g_1^2(x_2))$	$g_1^1(x_2)$
$p_1(g_1^2(x_2)) \cdot \overline{p}_1(g_3(g_1^2(x_2)))$	$g_2(x_1, g_1^1(x_2))$
$p_1(g_1^2(x_2)) \cdot p_1(g_3(g_1^2(x_2))) \cdot \overline{p}_1(g_3(g_3(g_1^2(x_2))))$	$g_2(x_1, g_2(x_1, g_1^1(x_2)))$
$\vdots \quad \vdots$	$\vdots \quad \vdots$

We can show that, in general, two schemata are equivalent if and only if their conditional expression lists agree in the sense illustrated by our demonstration of equivalence for S_2 and S_{13}. When the lists are finite the existence of a decision procedure is clear. At this time we do not know whether or not a decision procedure can be found for the general equivalence problem of persistent, commutative schemata.

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